

# Towards a Born term for hadrons

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We study bound states of abelian gauge theory in  $D = 1 + 1$  dimensions using an equal-time, Poincare-covariant framework. The normalization of the linear confining potential is determined by a boundary condition in the solution of Gauss' law for the instantaneous  $A^0$  field. As in the case of the Dirac equation, the norm of the relativistic fermion-antifermion ( $f\bar{f}$ ) wave functions gives inclusive particle densities. However, while the Dirac spectrum is known to be continuous we find that regular  $f\bar{f}$  solutions exist only for discrete bound state masses. The  $f\bar{f}$  wave functions are consistent with the parton picture when the kinetic energy of the fermions is large compared to the binding potential. We verify that the electromagnetic form factors of the bound states are gauge invariant and calculate the parton distributions from the transition form factors in the Bjorken limit. For relativistic states we find a large sea contribution at low  $x_{Bj}$ . Since the potential is independent of the gauge coupling the bound states may serve as “Born terms” in a perturbative expansion, in analogy to the usual plane wave *in* and *out* states.

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## I. INTRODUCTION

The hadron spectrum is simpler than one would expect. Deep inelastic scattering shows important contributions from sea quarks and gluons, yet  $q\bar{q}$  mesons and  $qqq$  baryons are successfully classified [1] in terms of only their valence quark degrees of freedom. Dynamical features such as masses and magnetic moments are consistent with the non-relativistic quark model, even though (light) quarks are known to be ultrarelativistic.

The situation is made even more interesting by the fact that relativistic bound states are generally difficult to address in field theory. The perturbative expansion, which works well for non-relativistic atoms in QED, appears to be excluded for QCD hadrons, notwithstanding the apparent similarities of hadronic and atomic spectra. The velocities of atomic constituents scale with the fine-structure constant,  $v \simeq \alpha$ . Thus relativistic dynamics ( $v \simeq 1$ ) appears to imply strong coupling ( $\alpha \gtrsim 1$ ). In the absence of a small expansion parameter analytic studies of relativistic bound states are generally model dependent.

A small (perturbative) coupling can nevertheless be compatible with a relativistic bound state. A simple example is provided by an electron bound in a strong external field. The strength of the field  $\propto Ze$  provides a new parameter: We may in principle have  $Ze$  of  $\mathcal{O}(1)$  while  $e \ll 1$ . To all orders in  $Ze$  the bound states are determined by diagrams where the electron interacts only with the external field. At this level the spectrum is exactly given by the Dirac equation. Self-interactions of the electron are  $\mathcal{O}(e^2)$  QED loop corrections, to be added perturbatively.

The Dirac bound states do include effects of virtual  $e^+e^-$  pairs which are created in the external field. This is made explicit by time-ordering the electron propagators. When the electron scatters into a negative energy state it propagates backward for some time, creating a “ $Z$ ” diagram. The intermediate state then contains an additional  $e^+e^-$  pair. The Klein paradox [2] demonstrated early on that the Dirac wave function cannot be interpreted as a one-particle probability density. The paradox is elegantly resolved by considering pair production effects using the methods of quantum field theory [3].

The Dirac spectrum can be classified in terms of a single electron degree of freedom [4]. This demonstrates that the virtual pairs present in relativistic bound states need not be explicitly seen in their spectra, in apparent analogy

to the situation for hadrons mentioned above. The Dirac states allow to examine this interesting phenomenon and to apply it to states that are bound by mutual interactions between the constituents rather than by an external potential [5]. Briefly, the argument is as follows.

The energy component of an electron's four-momentum  $p = (p^0, \mathbf{p})$  is unaffected by scattering in a time-independent (static) external potential  $A^0(\mathbf{x})$ . If  $p^0 > -m_e$  initially then  $p^0 \neq -\sqrt{\mathbf{p}^2 + m_e^2}$  throughout the scattering. Hence the  $i\varepsilon$  prescription at the negative energy pole of the electron propagators in any Feynman diagram is irrelevant. In particular, we may use retarded electron propagators  $\propto (p^0 + \sqrt{\mathbf{p}^2 + m_e^2} + i\varepsilon)^{-1}$  without affecting the positions of the bound-state poles in the Green function, *i.e.*, the spectrum. The *time-ordered* dynamics, however, will look very different: the electron propagates forward only, eliminating the  $Z$  diagrams and the *explicit*  $e^+e^-$  pairs. Only single electron Fock states are present at any time, distributed according to the standard Dirac wave function. Note that the change in  $i\varepsilon$  prescription is allowed only in the absence of loop corrections, *i.e.*, only at the level of the Dirac states without  $\mathcal{O}(e^2)$  perturbative corrections.

It is useful to formulate the above observation in the language of Hamiltonian field theory. If  $H$  is the QED Hamiltonian, where  $A = (A^0, \mathbf{A} = 0)$  is given by the static external potential, the Dirac bound-state condition

$$H |M\rangle = M |M\rangle \quad (1.1)$$

has no simple solution in terms of standard Fock states built on the perturbative vacuum  $|0\rangle$ . This is because  $H |0\rangle \neq 0$ , *i.e.*, the Hamiltonian creates pairs in the vacuum. Consequently the eigenstates  $|M\rangle$  must have Fock components with an unlimited number of  $e^+e^-$  pairs, as we already concluded above. In order to get retarded (rather than Feynman) electron propagators we need to change the boundary conditions, *i.e.*, the vacuum state  $|0\rangle \rightarrow |0\rangle_R$  such that the electron field operator annihilates the “retarded vacuum”,

$$\psi(x) |0\rangle_R = 0. \quad (1.2)$$

This eliminates pair production from the vacuum,  $H |0\rangle_R = 0$ . The eigenvalue condition (1.1) is then satisfied by the (apparent) single-electron state

$$|M\rangle = \int d\mathbf{x} \psi^\dagger(t=0, \mathbf{x}) \begin{bmatrix} \varphi(\mathbf{x}) \\ \chi(\mathbf{x}) \end{bmatrix} |0\rangle_R \quad (1.3)$$

provided the  $c$ -numbered wave function  $[\varphi, \chi]^T$  satisfies the Dirac equation

$$[-i\nabla \cdot \boldsymbol{\gamma} + e\gamma^0 A^0(\mathbf{x}) + m] \begin{bmatrix} \varphi(\mathbf{x}) \\ \chi(\mathbf{x}) \end{bmatrix} = M\gamma^0 \begin{bmatrix} \varphi(\mathbf{x}) \\ \chi(\mathbf{x}) \end{bmatrix}. \quad (1.4)$$

Since the Dirac state (1.3) is built on the retarded vacuum it has both positive energy components  $b^\dagger |0\rangle_R$  and negative energy components  $d |0\rangle_R$ . It was observed previously [6] that scattering amplitudes defined using retarded boundary conditions give *inclusive* cross sections. Analogously, the norm of the Dirac wave function may be viewed as an inclusive electron density.

Generalizing the Dirac state (1.3) to a bound state formed by two fermions we get [5]

$$|P\rangle = \int d\mathbf{x}_1 d\mathbf{x}_2 \bar{\psi}_1(t=0, \mathbf{x}_1) \exp[i\mathbf{P} \cdot (\mathbf{x}_1 + \mathbf{x}_2)/2] \Psi(\mathbf{x}_1 - \mathbf{x}_2) \psi_2(t=0, \mathbf{x}_2) |0\rangle_R, \quad (1.5)$$

where the wave function  $\Psi$  now has  $4 \times 4$  Dirac components. We assume that the fermions have distinct flavors,  $f = 1, 2$ , and define the boundary conditions such that  $\psi_1(x) |0\rangle_R = \psi_2^\dagger(x) |0\rangle_R = 0$ . The binding should be generated by the mutual interactions of the fermions in order for the state to be translationally invariant, *i.e.*, to have a well-defined four-momentum  $P = (E, \mathbf{P})$ . In analogy with the Dirac case we expect that relativistic binding at small (perturbative) gauge coupling can occur through an  $\mathcal{O}(e^0)$  contribution to the  $A^0$  component of the gauge field. This is achieved by imposing a non-vanishing boundary condition on the solution of the equations of motion,

$$\lim_{|\mathbf{x}| \rightarrow \infty} F_{\mu\nu} F^{\mu\nu}(\mathbf{x}) = -2\Lambda^4, \quad (1.6)$$

where  $\Lambda$  is a universal constant. Requiring  $\lim_{|\mathbf{x}| \rightarrow \infty} (\nabla A^0)^2 = \Lambda^4$  in Gauss' law adds an  $\mathcal{O}(e\Lambda^2)$  instantaneous interaction term to the Hamiltonian,  $H \rightarrow H + H_\Lambda$ , which for neutral states is effectively [5]

$$H_\Lambda = -\frac{e\Lambda^2}{4} \sum_{f,f'} \int d\mathbf{x} d\mathbf{y} \psi_f^\dagger \psi_f(t, \mathbf{x}) |\mathbf{x} - \mathbf{y}| \psi_{f'}^\dagger \psi_{f'}(t, \mathbf{y}), \quad (1.7)$$

and thus is leading compared to the  $\mathcal{O}(e^2)$  perturbative interactions. The condition of stationary time development at  $\mathcal{O}(e\Lambda^2)$ ,

$$(H_0 + H_\Lambda) |P\rangle = E |P\rangle, \quad (1.8)$$

is satisfied by the  $f_1 \bar{f}_2$  state (1.5) provided its wave function satisfies the bound-state equation

$$i\nabla_x \cdot \{\gamma^0 \gamma, \Psi(\mathbf{x})\} - \frac{1}{2} \mathbf{P} \cdot [\gamma^0 \gamma, \Psi(\mathbf{x})] + m_1 \gamma^0 \Psi(\mathbf{x}) - m_2 \Psi(\mathbf{x}) \gamma^0 = [E - V(\mathbf{x})] \Psi(\mathbf{x}), \quad (1.9)$$

with the linear potential

$$V(\mathbf{x}) = \frac{1}{2} e \Lambda^2 |\mathbf{x}|. \quad (1.10)$$

The wave functions  $\Psi$  and the energy eigenvalues  $E$  that solve the bound-state equation (1.9) depend on the 3-momentum  $\mathbf{P}$ . In this equal-time, Hamiltonian formalism boost invariance (as well as time translation invariance) is a dynamical symmetry. Because the field theory (here QED) is Poincaré invariant and is solved at lowest order in the coupling  $e$ , with the Poincaré invariant boundary condition (1.6), we expect the state (1.5) to be covariant. In [7] we verified this in  $D = 1 + 1$  dimensions using the boost generator  $M^{01}$ . The boosted state satisfies the bound-state equation (1.9) with the appropriately shifted momentum, *i.e.*,

$$(1 - id\xi M^{01}) |P^0, P^1\rangle = |P^0 + d\xi P^1, P^1 + d\xi P^0\rangle. \quad (1.11)$$

Also in  $D = 3 + 1$  dimensions the energy eigenvalues of (1.9) have the required dependence on the momentum [8],

$$E = \sqrt{M^2 + \mathbf{P}^2}. \quad (1.12)$$

This indicates that the states are Poincaré covariant also in four-dimensional space-time.

In order to keep the discussion as simple as possible we use the example of an abelian gauge theory (QED). The relevant application of this method is to hadrons in QCD, where a linear potential is desired. As shown in [5, 7] the generalization to a non-abelian symmetry is straightforward.

The present paper is a sequel to our study [7] of Poincaré invariance. Here we examine other features of the wave functions in  $D = 1 + 1$ . They have several unusual properties, some of which are shared with the (previously known) solutions of the Dirac equation. The  $D = 1 + 1$  wave functions can be expressed in terms of confluent hypergeometric functions, which allows detailed analytic and numerical studies. We expect that the results in this paper shed some light also on the properties of the solutions of (1.9) in  $D = 3 + 1$ .

In the next section we study the solutions of the Dirac equation (1.4) for a linear potential in  $D = 1 + 1$  dimensions. We show how, for nearly non-relativistic dynamics ( $m \gtrsim Ze$ ), the solutions agree with those of the corresponding Schrödinger equation at small fermion separations  $|x|$ , but reflect pair production at separations where  $V(x) \gtrsim 2m$ . In Sec. III we find the analytic solutions of the  $f\bar{f}$  bound-state equation (1.9) in  $D = 1 + 1$  dimensions. Similarly to the Dirac wave functions they are not square integrable since their norm tends to a constant at large  $|x|$ . The  $f\bar{f}$  wave functions, however, are generally singular at  $V(x) = E \pm P^1$ . Solutions that are regular at these points<sup>1</sup> exist only for discrete bound state masses [14]. This differs qualitatively from the Dirac equation, whose solutions are regular at all  $x$ , giving a continuous mass spectrum. In Sec. IV we show that the normalization of the wave function at  $x = 0$  can be determined by requiring duality between the bound state ( $\delta$  distribution) and fermion-loop contributions to the imaginary parts of current propagators. A self-consistent normalization is obtained in all Lorentz frames and for all currents. For highly excited bound states the wave functions [at low  $|x|$ , hence small  $V(x)$ ] turn into plane waves of positive energy fermions as expected in the parton model. In Sec. V we evaluate the electromagnetic form factors and show that they are gauge invariant (in any dimension). In Sec. VI we express Deep Inelastic Scattering in terms of transition form factors in the Bjorken limit. For relativistic states (small fermion masses) the  $D = 1 + 1$  parton distributions grow large at small  $x_{Bj}$ , with  $x_{Bj} f(x_{Bj}) \propto \log^2(x_{Bj})$ . Our conclusions are given in Sec. VII.

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<sup>1</sup> When the fermion masses are unequal,  $m_1 \neq m_2$  in (1.9), the wave functions are fully regular only in the Infinite Momentum Frame,  $P^1 \rightarrow \infty$ .

## II. DIRAC EQUATION IN $D = 1 + 1$

Some of the novel properties of the  $f\bar{f}$  states that we study in  $D = 1 + 1$  dimensions, notably the asymptotically constant norm of their wave functions, are shared by electrons bound in an external linear  $A^0$  potential. Soon after Dirac first proposed his wave equation [9] it was realized [10–12] that solutions of the Dirac equation generally cannot be normalized, in contrast to the case of the Schrödinger equation, where the requirement of a finite normalization integral leads to the quantization of the energy spectrum. Thus in [12] it was shown that the solutions of the Dirac equation in  $D = 1 + 1$  have a constant norm as  $|x| \rightarrow \infty$  for all potentials of the form  $A^0(x) = x^n$  (or combinations thereof), where  $n \neq 0$  is any positive or negative integer. A similar result holds in  $D = 3 + 1$  dimensions when  $\mathbf{A} = 0$ , for central potentials  $A^0(r)$  which are polynomials in  $r$  or in  $1/r$ , with the interesting exception of  $A^0(r) \propto 1/r$  [12]. The Dirac energy spectrum is thus generally continuous, and the completeness relation involves a continuous set of eigenfunctions [13].

The constant norm of the Dirac wave function reflects pair production in a strong potential. The phenomenon is related to the Klein paradox [2], which requires a multiparticle framework for its resolution [3]. The pairs can be seen to arise from  $Z$  diagrams when the scattering of an electron in an external field is time-ordered. As already mentioned in the Introduction, in the case of static  $A^0$  potentials the same spectrum is obtained with retarded electron propagators [5]. Then electrons of both positive and negative energies propagate forward in time, and the bound states are described by single particle Dirac wave functions. Due to the retarded boundary conditions the norm of the Dirac wave function is, however, “inclusive” in nature, in analogy with the more familiar concept of inclusive scattering cross sections [6].

The analytic solution of the Dirac equation (1.4) in  $D = 1 + 1$  for a linear potential was first given in [11]. We include the derivation below for completeness and to introduce our notation. We also discuss some numerical properties of the solutions, which appear not to be widely known.

### A. General solution

We use a standard 2-dimensional representation of the Dirac matrices in terms of Pauli matrices,

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2, \quad \gamma^0\gamma^1 = \gamma_5 = \sigma_1. \quad (2.1)$$

In QED<sub>2</sub> the potential generated by a static source of charge  $Ze$  is  $V(x) = eA^0(x) = \frac{1}{2}Ze^2|x|$ . We use units where  $Ze^2 = 1$ , hence

$$V(x) = \frac{1}{2}|x|. \quad (2.2)$$

In the following all dimensionful quantities can be given their physical dimensions through multiplication by the appropriate power of  $Ze^2$ .

The Dirac equation (1.4) implies

$$\begin{aligned} -i\partial_x\chi &= (M - V - m)\varphi, \\ -i\partial_x\varphi &= (M - V + m)\chi. \end{aligned} \quad (2.3)$$

We choose the phases such that  $\varphi(x)$  is real and  $\chi(x)$  is imaginary. This means that the solutions are characterized by two real parameters, *e.g.*,  $\varphi(0)$  and  $-i\chi(0)$ . By adding/subtracting the equations at  $x$  and  $-x$  we may impose that

$$\varphi(x) = \eta \varphi(-x), \quad \chi(x) = -\eta \chi(-x), \quad (2.4)$$

where  $\eta = \pm 1$ . This allows us to consider solutions in the region  $x \geq 0$  only. Continuity at  $x = 0$  requires  $\chi(0) = 0$  for  $\eta = +1$ , and  $\varphi(0) = 0$  for  $\eta = -1$ . The equations (2.3) then ensure that  $\partial_x\varphi(0) = 0$  for  $\eta = 1$  and  $\partial_x\chi(0) = 0$  for  $\eta = -1$ .

The two first-order equations (2.3) give rise to a second-order equation for  $\varphi(x)$ ,

$$\partial_x^2\varphi(x) + \frac{\varepsilon(x)}{2(M - V + m)}\partial_x\varphi(x) + [(M - V)^2 - m^2]\varphi(x) = 0, \quad (2.5)$$

where  $\varepsilon(x) \equiv x/|x|$  is the sign function. For  $x \rightarrow \infty$  the term  $V^2\varphi(x)$  must be balanced by  $\partial_x^2\varphi(x)$ , hence

$$\varphi(x \rightarrow \infty) \sim \exp(\pm ix^2/4). \quad (2.6)$$

From (2.3) follows that  $\chi(x)$  has a similar asymptotic behavior. Since the norms  $|\varphi(x)|$  and  $|\chi(x)|$  tend to constants for  $x \rightarrow \infty$  the Dirac wave functions are not normalizable [10–12], unlike the solutions of the non-relativistic Schrödinger equation. As we shall see, the solutions have features which support the interpretation that their norm at large  $V(x)$  reflects virtual pair contributions.

The coefficient of  $\partial_x \varphi(x)$  in (2.5) is singular at  $M - V + m = 0$ . Assuming  $\varphi(x) \sim (M - V + m)^\alpha$  as  $M - V + m \rightarrow 0$  we find  $\alpha = 0$  or  $2$ . Hence the general solutions  $\varphi(x)$  and  $\chi(x)$  have no singularities at finite  $x$ . Since the wave functions are not square integrable there is no restriction on the eigenvalues  $M$ . In Sec. II B we discuss how the discrete eigenvalues required by the Schrödinger equation emerge nevertheless in the non-relativistic domain ( $m \gg 1$ ). In Sec. II C we show that any two solutions with different eigenvalues  $M$  are orthogonal.

Since only the combination  $M - V(x)$  appears in the Dirac equation (2.3) it is convenient to replace  $x$  by the variable<sup>2</sup>

$$\sigma = (M - V)^2, \quad \partial_x = \frac{d\sigma}{dx} \partial_\sigma = -(M - V) \partial_\sigma. \quad (2.7)$$

The Dirac equation then reads<sup>3</sup>, in the region where  $M - V(x) > 0$ ,

$$\begin{aligned} i\partial_\sigma \chi(\sigma) &= \left(1 - \frac{m}{\sqrt{\sigma}}\right) \varphi(\sigma), \\ i\partial_\sigma \varphi(\sigma) &= \left(1 + \frac{m}{\sqrt{\sigma}}\right) \chi(\sigma). \end{aligned} \quad (2.8)$$

We may combine  $\varphi$  and  $\chi$  into the single complex function

$$\phi(\sigma) \equiv [\varphi(\sigma) + \chi(\sigma)] e^{i\sigma} \quad \text{and conversely} \quad \varphi(\sigma) = \text{Re}[\phi(\sigma) e^{-i\sigma}], \quad \chi(\sigma) = i \text{Im}[\phi(\sigma) e^{-i\sigma}]. \quad (2.9)$$

The second-order equation for  $\phi(\sigma)$  is then

$$2\sigma \partial_\sigma^2 \phi + (1 - 4i\sigma) \partial_\sigma \phi - 2m^2 \phi = 0, \quad (2.10)$$

with the general solution

$$\phi(\sigma) = (a_1 + ib_1) {}_1F_1\left(-\frac{1}{2}im^2, \frac{1}{2}, 2i\sigma\right) + (a_2 + ib_2) \sqrt{\sigma} {}_1F_1\left(\frac{1}{2} - \frac{1}{2}im^2, \frac{3}{2}, 2i\sigma\right), \quad (2.11)$$

where  ${}_1F_1$  is the confluent hypergeometric function and the  $a_i, b_i$  are real constants. From (2.9) we find

$$\begin{aligned} \varphi(\sigma \rightarrow 0) &= a_1 + a_2 \sqrt{\sigma} + \mathcal{O}(\sigma), \\ \chi(\sigma \rightarrow 0) &= ib_1 + ib_2 \sqrt{\sigma} + \mathcal{O}(\sigma). \end{aligned} \quad (2.12)$$

Matching the terms of  $\mathcal{O}(1/\sqrt{\sigma})$  in (2.8) gives the relations

$$b_2 = 2ma_1, \quad a_2 = 2mb_1. \quad (2.13)$$

The general solution of the  $D = 1 + 1$  Dirac equation (2.3) with the linear potential (2.2) of QED<sub>2</sub> is thus given by

$$\psi(\sigma) \equiv \varphi(\sigma) + \chi(\sigma) = \left[ (a + ib) {}_1F_1\left(-\frac{im^2}{2}, \frac{1}{2}, 2i\sigma\right) + (b + ia) 2m \varepsilon(M - V) \sqrt{\sigma} {}_1F_1\left(\frac{1 - im^2}{2}, \frac{3}{2}, 2i\sigma\right) \right] \exp(-i\sigma), \quad (2.14)$$

where  $a$  and  $b$  are real parameters and  $\sigma = (M - V)^2$ .  $\varphi$  and  $\chi$  are given by the real and imaginary parts of the *rhs.*, respectively. The sign function  $\varepsilon(M - V)$  ensures that the solution is valid also in the region where  $M - V(x) < 0$ , since  $\sqrt{\sigma}$  in (2.8) changes sign at  $\sigma = 0$ . This solution agrees with Eq. (14) of [11].

The solution (2.14) is valid for  $x \geq 0$ . The wave functions for  $x < 0$  are given by the symmetry relations (2.4). Continuity at  $x = 0$  requires that the anti-symmetric (real or imaginary) part of the wave function vanishes at  $x = 0$ , which also ensures that the  $x$  derivative of the symmetric part vanishes. The positions  $\sigma = \sigma_0$  where either the real

<sup>2</sup> We consider solutions only for  $x \geq 0$  in the following. The parity condition (2.4) gives the solutions for  $x < 0$ .

<sup>3</sup> Here and in the following we use the shorthand notation  $\varphi(\sigma) \equiv \varphi[\sigma(x)]$ , and similarly for  $\chi(x)$ .

or imaginary part of the *rhs.* in (2.14) vanishes determine the mass eigenvalues  $M = \sqrt{\sigma_0}$ , since continuity allows to set  $x = 0$  at  $\sigma_0$ . The eigenvalues  $M$  thus depend on the ratio  $a/b$  of the parameters.

The asymptotic behavior of the wave function at large  $x$ ,

$$\begin{aligned} \lim_{x \rightarrow \infty} \psi(\sigma) = & \sqrt{\pi} e^{\pi m^2/4} \left[ \frac{a + ib}{\Gamma[\frac{1}{2}(1 + im^2)]} + \frac{a - ib}{\Gamma(1 + \frac{1}{2}im^2)} \frac{me^{-i\pi/4}}{\sqrt{2}} \right] (2\sigma)^{im^2/2} e^{-i\sigma} \\ & + \frac{\sqrt{\pi}}{\sqrt{2}\sigma} e^{\pi(m^2-i)/4} \left[ \frac{a + ib}{\Gamma(-\frac{1}{2}im^2)} - \frac{a - ib}{\Gamma[\frac{1}{2}(1 - im^2)]} \frac{me^{i\pi/4}}{\sqrt{2}} \right] (2\sigma)^{-im^2/2} e^{i\sigma} + \mathcal{O}\left(\frac{1}{\sigma}\right), \end{aligned} \quad (2.15)$$

oscillates with  $\sigma \simeq x^2/4$  in agreement with the general result (2.6). Hence the norm of the wave function tends to a constant at high  $x$ . Unlike in the non-relativistic limit (the Schrödinger equation), the parameters of the solution cannot be determined by a normalizability condition.

### B. The non-relativistic limit

The fact that the eigenvalues  $M$  of the Dirac equation (2.3) depend on the ratio  $a/b$  of the parameters in (2.14) raises the question about the approach to the non-relativistic limit. In  $D = 1 + 1$  dimensions for a fixed potential this limit is equivalent to taking  $m \rightarrow \infty$ , scaling simultaneously coordinates and momenta appropriately. For a linear potential the scaling of the coordinate and the binding energy  $E_b = M - m$  is

$$x \sim E_b \sim m^{-1/3}. \quad (2.16)$$

In the non-relativistic limit the Dirac equation reduces to the Schrödinger equation

$$-\frac{1}{2m} \partial_x^2 \rho(x) + \frac{1}{2} |x| \rho(x) = E_b \rho(x), \quad (2.17)$$

whose normalizable solutions are given by the Airy function,

$$\rho(x) = N \text{Ai}[m^{1/3}(x - 2E_b)] \quad (x > 0), \quad (2.18)$$

with  $\rho(-x) = \pm \rho(x)$ . The solutions are differentiable at  $x = 0$  only for discrete binding energies. How are the same eigenvalues selected in the Dirac equation at large  $m$ ?

It turns out that the two independent solutions in (2.14) become degenerate at large  $m$ . Setting  $M = m + E_b$  in (2.7) and noting the scaling (2.16) we have

$$\sigma = [m^2 + m(2E_b - |x|)] + \mathcal{O}(m^{-2/3}) \quad (2.19)$$

At large  $m$  we may use a stationary-phase approximation in the integral representation of the Hypergeometric functions in (2.14), giving

$$\psi(\sigma) = (1 + i)(a + b) \sqrt{\pi} m^{1/3} e^{\pi m^2/2 + i(m^2 - \pi/4)} \text{Ai}[m^{1/3}(|x| - 2E_b)] \left[ 1 + \mathcal{O}(m^{-2/3}) \right], \quad (2.20)$$

which agrees with the standard solution (2.18) of the Schrödinger equation. Consequently the non-relativistic limit of the general solution (2.14) does not depend on the ratio  $a/b$ .

As seen from Fig. 1 the Dirac eigenvalues are insensitive to  $a/b$  already for  $m \gtrsim 1$ , where they merge with the bound-state mass given by the Schrödinger equation. At large  $m$  there is a very narrow range of  $a/b$  where the (real or imaginary parts of the) two terms on the *rhs.* of (2.14) nearly cancel. Only in this range does the position of the zero,  $\sigma = \sigma_0$ , depend on the precise value of  $a/b$ . For, *e.g.*,  $m = 2.5$  this occurs for the ground state near  $a/b = -1.041$ , and the width of the interval in  $a/b$  that gives a continuum range of masses  $M$  is of  $\mathcal{O}(10^{-6})$ . The approximate cancellation in (2.14) then makes the wave function grow very rapidly near  $\sigma = \sigma_0$ , mimicking the exponential growth of the Schrödinger solutions for general  $M$ . For generic values of  $a/b$  and  $m \gtrsim 1$  the bound-state masses given by the Dirac equation agree with those of the normalizable solutions to the Schrödinger equation as indicated in Fig. 1.

The issue of the approach to non-relativistic dynamics was also addressed in [13]. A measure of the relativistic effects was provided by the distance from the real axis of certain poles related to the Dirac eigenvalues, which was found to be  $\sim \exp(-\kappa m^2)$ , with  $\pi \leq \kappa \leq 2\pi$ . In view of this it is interesting to note that the (typical) difference

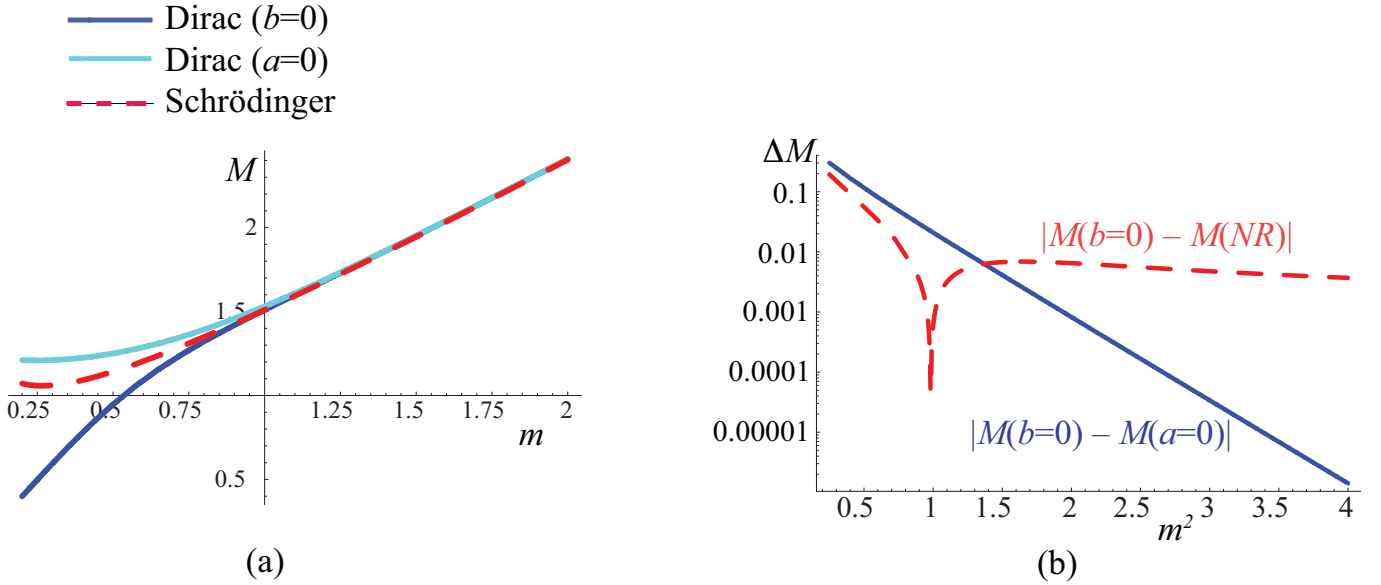


FIG. 1: Ground state masses  $M$  as a function of the constituent fermion mass  $m$  (in units of  $\sqrt{Ze^2}$ ). (a) The dark (light) blue solid curves correspond to the Dirac wave function (2.14) with  $b = 0$  ( $a = 0$ ). The dashed red curve shows  $M = m + E_b$ , with  $E_b$  the eigenvalue given by the Schrödinger equation (2.17) (which is reliable for  $m \gtrsim 1$  only). (b) Absolute values of the ground state mass differences plotted versus  $m^2$  on a logarithmic scale. The mass difference between the Dirac solutions for  $b = 0$  and  $a = 0$  (solid blue line) decreases exponentially with  $m^2$ , much faster than the difference of the  $b = 0$  and Schrödinger masses (dashed red curve).

between the Dirac eigenvalues obtained with different  $a/b$  decreases similarly with  $m$  (here  $\kappa = \pi$ ), as shown by the solid (blue) line in Fig. 1(b).

As expected, the Dirac wave function is similar to the Schrödinger one only for values of  $x$  such that  $V(x) \ll m$ , *i.e.*, for weak binding. Thus the (upper component of) the Dirac wave function agrees with the Schrödinger wave function in the non-relativistic regime, as seen in Fig. 2 (where  $m = 2.5$  and  $b = 0$ ). Following its decrease to very small values the Dirac wave function begins to increase at a value of  $x$  where  $V(x) \simeq M$ , and becomes  $\mathcal{O}(1)$  again, initiating its asymptotic oscillations (2.6) when  $V(x) \simeq 2M$ . This is because the wave function depends on  $x$  through the variable  $\sigma = [M - V(x)]^2$ , which takes the same value at  $x = 0$  as at  $V = 2M$ . The start of the oscillations at a potential energy corresponding to pair production is indicative of the relation between the non-vanishing asymptotic norm and multiparticle effects (the  $Z$  diagrams mentioned above).

### C. Orthogonality

Two distinct solutions of the Dirac equation (2.3),  $\Phi_k = (\varphi_k \ \chi_k)^T$  and  $\Phi_\ell$  with eigenvalues  $M_k \neq M_\ell$  satisfy

$$\begin{aligned} -i\sigma_1 \partial_x \Phi_k + m\sigma_3 \Phi_k &= (M_k - V)\Phi_k, \\ i\partial_x \Phi_\ell^\dagger \sigma_1 + m\Phi_\ell^\dagger \sigma_3 &= (M_\ell - V)\Phi_\ell^\dagger. \end{aligned} \quad (2.21)$$

Multiplying the first equation by  $\Phi_\ell^\dagger$  from the left and the second by  $\Phi_k$  from the right and subtracting, we find

$$-i\partial_x (\Phi_\ell^\dagger \sigma_1 \Phi_k) = (M_k - M_\ell) (\Phi_\ell^\dagger \Phi_k). \quad (2.22)$$

In terms of the solution  $\psi(x) = \varphi(x) + \chi(x)$  in (2.14) this is

$$\partial_x [\text{Im}(\psi_\ell^* \psi_k)] = (M_k - M_\ell) \text{Re}(\psi_\ell^* \psi_k). \quad (2.23)$$

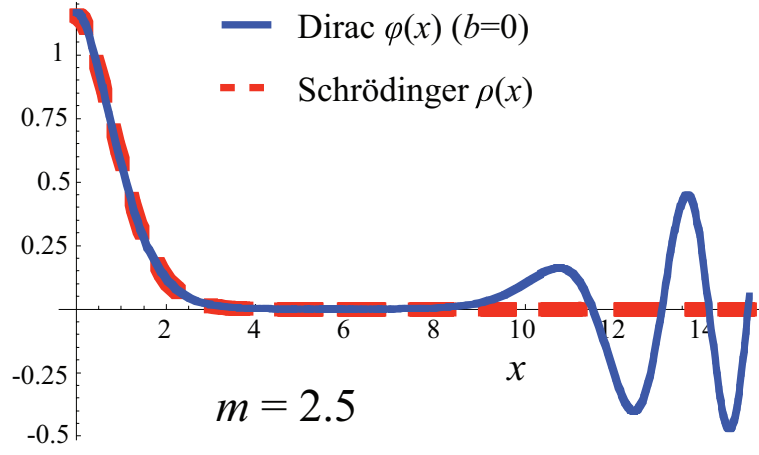


FIG. 2: The upper component  $\varphi(x)$  of the Dirac wave function (continuous blue line) and the Schrödinger wave function  $\rho(x)$  of (2.18) (dashed red line). The fermion mass is  $m = 2.5$  and the parameter  $b = 0$  in the analytic solution (2.14). Both solutions are normalized to unity in the range  $0 \leq x \leq 6$ .

Recalling that  $\varphi(x) = \eta \varphi(-x)$  is real and  $\chi(x) = -\eta \chi(-x)$  is imaginary both sides of (2.23) are odd functions of  $x$  if  $\eta_k = -\eta_\ell$ . Since wave functions of opposite parity are trivially orthogonal it suffices to consider the case  $\eta_k = \eta_\ell$ . Integrating both sides from  $x = 0$  to  $x = X$  and noting that the insertion at  $x = 0$  of the *lhs.* vanishes we find

$$\text{Im}[\psi_\ell^* \psi_k(x = X)] = (M_k - M_\ell) \int_0^X dx \text{Re}[\psi_\ell^* \psi_k(x)]. \quad (2.24)$$

The leading term in the asymptotic limit of (2.15) is of the form

$$\psi(x \rightarrow +\infty) = \eta \psi^*(x \rightarrow -\infty) = C \sigma^{im^2/2} e^{-i\sigma}, \quad (2.25)$$

where  $\sigma = M^2 - Mx + x^2/4$ , and the complex constant  $C$  depends on  $m$  as well as on  $a$  and  $b$ , which need not be the same for each bound-state level. For a sufficiently large  $X$  that the asymptotic form (2.25) of the wave functions applies we have

$$\psi_\ell^* \psi_k(x = X) \simeq C_{k\ell} e^{i(M_k - M_\ell)X}, \quad (2.26)$$

where  $C_{k\ell} = C_\ell^* C_k e^{i(M_\ell^2 - M_k^2)}$ . If (to simplify the discussion) we choose  $X$  such that<sup>4</sup>

$$\cos[(M_k - M_\ell)X] = 0 \quad (2.27)$$

(2.24) gives, taking into account that the integrand on the *rhs.* is symmetric in  $x$ ,

$$\int_{-X}^X dx \text{Re}[\psi_\ell^* \psi_k(x)] \simeq \frac{2\text{Re}(C_{k\ell}) \sin[(M_k - M_\ell)X]}{M_k - M_\ell}. \quad (2.28)$$

We recognize the *rhs.* as a standard representation of the  $\delta$  distribution. Taking  $X \rightarrow \infty$  we have

$$\int_{-\infty}^{\infty} dx \Phi_\ell^\dagger \Phi_k = \int_{-\infty}^{\infty} dx \text{Re}[\psi_\ell^* \psi_k(x)] = \text{Re}(C_{k\ell}) 2\pi \delta(M_k - M_\ell). \quad (2.29)$$

The freedom in choosing the parameter  $a/b$  in the general solution (2.14) allows bound-state solutions for a continuous range of masses  $M$ . Since all these solutions are orthogonal a completeness sum must include them all, as shown in [13]. The spectrum of the Dirac equation with a linear potential is thus continuous, similar to the case of non-interacting plane waves.

<sup>4</sup> Relaxing this assumption leads to an extra term in Eq. (2.28), which is singular at  $M_k = M_\ell$  but does not contribute to the final result.



### III. SOLUTIONS OF THE TWO-FERMION BOUND-STATE EQUATION IN $D = 1 + 1$

In this Section we give the general solution for the wave functions  $\Psi = e^{i\varphi}\Phi$  that describe the bound states (1.5) of two fermions in  $D = 1 + 1$  dimensions,

$$|E, P\rangle \equiv \int dx_1 dx_2 \exp\left[\frac{1}{2}iP(x_1 + x_2)\right] \bar{\psi}_1(0, x_1) e^{i\varphi} \Phi(x_1 - x_2) \psi_2(0, x_2) |0\rangle_R. \quad (3.1)$$

The 2-momentum of the bound state is denoted  $(E, P)$ , and the Dirac matrices are defined as in (2.1). The bound-state equation (1.9) is then [5, 7]

$$i\partial_x \{\sigma_1, \Phi(x)\} - (\partial_x \varphi) \{\sigma_1, \Phi(x)\} - \frac{1}{2}P[\sigma_1, \Phi(x)] + m_1\sigma_3\Phi(x) - m_2\Phi(x)\sigma_3 = [E - V(x)]\Phi(x). \quad (3.2)$$

The linear potential imposed by the boundary condition (1.6) is of the form (1.10). We take the coefficient of  $\frac{1}{2}|x|$  as our energy scale and thus have the same potential (2.2) as in the Dirac equation, now with  $x = x_1 - x_2$ .

We showed in [7] that  $E = \sqrt{P^2 + M^2}$  and worked out the  $P$  dependence of the wave function. It turned out to be convenient to introduce the variable

$$\sigma \equiv (E - V)^2 - P^2 = M^2 - 2EV + V^2 \equiv \Pi^2, \quad (3.3)$$

which in the rest frame ( $P = 0$ ) coincides with the variable  $\sigma$  defined in (2.7), which we used in the Dirac equation<sup>5</sup>. The expression for  $\sigma$  suggests to define<sup>6</sup> the “kinematical 2-momentum”  $\Pi^\mu = P^\mu - eA^\mu$ , where  $P^\mu = (E, P)^\mu$  is the bound-state momentum and  $eA^\mu = (V, 0)^\mu$ ,

$$\Pi(x) = (E - V(x), P) \equiv (\cosh \zeta, \sinh \zeta)\sqrt{\sigma}. \quad (3.4)$$

Here the last equality holds when  $\sigma = \Pi^2 > 0$ . The variable  $\sigma(x) = \sigma(-x)$  first decreases with increasing  $x > 0$ , from  $\sigma = M^2$  down to  $\sigma = -P^2$  at  $V(x) = E$ , and then increases with  $x$ , behaving asymptotically as  $\sigma \simeq x^2/4$ .

The common phase  $\varphi(x)$ , which for convenience is extracted from the  $2 \times 2$  wave function  $\Phi$  in the state (3.1), is

$$\varphi(x) = \varepsilon(x)(m_1^2 - m_2^2)(\zeta - \xi) = \frac{1}{2}\varepsilon(x)(m_1^2 - m_2^2) \log \left[ \left( \frac{M}{E + P} \right)^2 \left| \frac{E - V + P}{E - V - P} \right| \right]. \quad (3.5)$$

Here  $\zeta$  is defined<sup>7</sup> in (3.4), and  $\xi$  is the rapidity of the bound state,  $e^\xi = (E + P)/M$ .

Taking the complex conjugate of the bound-state equation and changing  $x \rightarrow -x$  we see that  $\Phi(x)$  and  $\Phi^*(-x)$  satisfy the same equation. Consequently we may define solutions of definite parity  $\eta = \pm 1$  by

$$\Phi^\eta(x) = \Phi(x) + \eta \Phi^*(-x), \quad \Phi^\eta(-x) = \eta \Phi^{\eta*}(x). \quad (3.6)$$

In the following we first construct solutions for  $x \geq 0$  and then complete them to the region  $x < 0$  according to (3.6), requiring continuity at  $x = 0$ .

The general structure of a  $2 \times 2$  wave function  $\Phi(x)$  that satisfies (3.2) is [7]

$$\begin{aligned} \Phi(x) &= \phi + \frac{1}{\sigma}(m_1 \mathbb{I}^\dagger \phi - m_2 \phi \mathbb{I}^\dagger), \\ \phi(x) &\equiv \Phi_0(x) + \Phi_1(x)\sigma_1, \end{aligned} \quad (3.7)$$

where  $\Phi_0$  and  $\Phi_1$  are scalar functions of  $x$ . Inserting these expressions into the bound-state equation (3.2) and substituting the variable  $\sigma$  of (3.3) for  $x$  using

$$\partial_x = -(E - V)\partial_\sigma \quad (x \geq 0), \quad (3.8)$$

<sup>5</sup> The invariant mass  $M$  is now that of the  $f\bar{f}$  system. The present variable  $\sigma$  is related to the variable  $s$  of [7] through  $\sigma = M^2 - 2\varepsilon(x)s(x)$ .

<sup>6</sup> Here the space component of  $\Pi$  has the opposite sign compared to its definition in [7].

<sup>7</sup> Notice that (3.4) defines  $\zeta$  only for  $\sigma > 0$ . The last expression in (3.5) can be taken as the definition of  $\varphi$  for  $\sigma \leq 0$ , and this is enough to make the wave function  $\Phi$  well-defined.

we find that  $\Phi_0$  and  $\Phi_1$  satisfy

$$-2i\partial_\sigma\Phi_1(\sigma) = \left[1 - \frac{(m_1 - m_2)^2}{\sigma}\right]\Phi_0(\sigma), \quad -2i\partial_\sigma\Phi_0(\sigma) = \left[1 - \frac{(m_1 + m_2)^2}{\sigma}\right]\Phi_1(\sigma). \quad (3.9)$$

The explicit dependence on  $E$  and  $P$  has disappeared, which means that  $\Phi_0$  and  $\Phi_1$  are the same functions of  $\sigma$  in any frame. They are, however,  $P$  dependent when viewed as functions of  $x$  due to the relation (3.3) between  $\sigma$  and  $x$ . The full  $2 \times 2$  wave function  $\Phi$  is expressed in terms  $\Phi_0$  and  $\Phi_1$  by (3.7). The relation between the wave function in a frame where the CM momentum is  $P$  to the rest frame ( $P = 0$ ) wave function is given by

$$\Phi(\sigma) = e^{-\sigma_1\zeta/2}\Phi^{(P=0)}(\sigma)e^{\sigma_1\zeta/2}, \quad (3.10)$$

with  $\zeta$  defined by (3.4) and  $\sigma \geq 0$ .

#### A. $f\bar{f}$ solutions for $m_1 = m_2 = m$

We first consider the equal mass case,  $m_1 = m_2 = m$ . Then the phase  $\varphi = 0$  in (3.5) and the coupled equations (3.9) reduce to a second-order equation for  $\Phi_1(\sigma)$ , which has the form of a Coulomb wave equation,

$$4\partial_\sigma^2\Phi_1 + \left(1 - \frac{4m^2}{\sigma}\right)\Phi_1 = 0. \quad (3.11)$$

The solution will oscillate asymptotically,  $\Phi_1(\sigma \rightarrow \infty) \sim \exp(\pm i\sigma/2)$ , analogously to the behavior of the Dirac wave function (2.6). The general solution for  $\Phi_1$  is

$$\Phi_1(\sigma) = \sigma e^{-i\sigma/2} [a {}_1F_1(1 - im^2, 2, i\sigma) + b U(1 - im^2, 2, i\sigma)], \quad (3.12)$$

where  $a$  and  $b$  are constants and  $U(\alpha, \beta, z)$  is the Confluent Hypergeometric function of the second kind.

The behavior of the  $U$  function for small argument,

$$U(\alpha, 2, z \rightarrow 0) = \frac{1}{\Gamma(\alpha)} \frac{1}{z} + \mathcal{O}(\log z), \quad (3.13)$$

causes the  $2 \times 2$  wave function  $\Phi(x)$  in (3.7) to be singular at  $\sigma = 0$  if  $b \neq 0$  in (3.12): Then  $\lim_{\sigma \rightarrow 0} \Phi_1(\sigma) = -ib/\Gamma(1 - im^2)$  is non-vanishing, and the singular factor  $1/\sigma$  is uncanceled in (3.7). Such a singularity at  $\sigma = 0$  prevents even a *local* normalizability of the wave function, and causes the orthogonality integrals (3.35) (Sec. III C below) to diverge.

The  $f\bar{f}$  bound-state equation thus differs significantly from the Dirac equation (2.3), even though both wave functions are oscillatory at large  $|x|$ . The general solution for the Dirac wave function is regular for finite  $x$  and thus locally normalizable, whereas this is true for the  $f\bar{f}$  wave function only provided  $b = 0$  in (3.12).

If we express the bound state mass as  $M = 2m + E_b$  then in the limit of large fermion masses ( $m \rightarrow \infty$ ) the binding energy  $E_b$  and the coordinate  $x$  scale as in (2.16) (in the rest frame,  $P = 0$ ). Substituting  $\sigma \simeq 4m^2 + 2m(E_b - \frac{1}{2}|x|)$  in the solution (3.12) and using a stationary-phase approximation in the integral representation of the Hypergeometric functions they turn into solutions of the non-relativistic Schrödinger equation,

$$\begin{aligned} \sigma e^{-i\sigma/2} {}_1F_1(1 - im^2, 2, i\sigma) &= \left(\frac{2}{m}\right)^{2/3} e^{\pi m^2} Ai\left[\left(\frac{1}{2}m\right)^{1/3}(|x| - 2E_b)\right] \\ \sigma e^{-i\sigma/2} U(1 - im^2, 2, i\sigma) &= -(2m^2)^{2/3} \frac{\pi e^{-\pi m^2}}{\Gamma(1 - im^2)} \left\{ Ai\left[\left(\frac{1}{2}m\right)^{1/3}(|x| - 2E_b)\right] + i Bi\left[\left(\frac{1}{2}m\right)^{1/3}(|x| - 2E_b)\right] \right\}, \end{aligned} \quad (3.14)$$

up to  $\mathcal{O}(m^{-4/3})$  corrections. The result for the  $U$  function involves the non-normalizable Airy  $Bi$  function.

In order to ensure local normalizability<sup>8</sup>, orthogonality of the lowest-order solutions as well as the correct behavior in the non-relativistic limit we set  $b = 0$  and thus consider

$$\begin{aligned} \Phi_1(\sigma) &= N \sigma e^{-i\sigma/2} {}_1F_1(1 - im^2, 2, i\sigma) = \frac{N \sinh(\pi m^2)}{\pi m^2} \sigma e^{-i\sigma/2} \int_0^1 du e^{i\sigma u} u^{-im^2} (1 - u)^{im^2} = \Phi_1^*(\sigma), \\ \Phi_0(\sigma) &= -\Phi_1(\sigma) - 2iN e^{-i\sigma/2} {}_1F_1(1 - im^2, 1, i\sigma) = -\Phi_0^*(\sigma), \end{aligned} \quad (3.15)$$

<sup>8</sup> The requirement of local normalizability was previously used in [14].

where we assumed the normalization constant  $N$  to be real. This makes  $\Phi_1(\sigma)$  real for all  $\sigma$ , as may be seen by a  $u \rightarrow 1 - u$  transformation of its integral representation. Correspondingly,  $\Phi_0(\sigma)$  is purely imaginary according to (3.9).

The asymptotic behavior for large  $|\sigma|$  is

$$\begin{aligned}\Phi_1(\sigma \rightarrow \pm\infty) &\simeq \sqrt{\frac{2}{\pi}} \frac{N}{m} \sqrt{e^{2\pi m^2} - 1} e^{-\pi m^2 \theta(-\sigma)} \sin \left[ \frac{\sigma}{2} - m^2 \log(|\sigma|) + \arg \Gamma(1 + im^2) \right] [1 + \mathcal{O}(\sigma^{-1})], \\ \Phi_0(\sigma \rightarrow \pm\infty) &\simeq -i \sqrt{\frac{2}{\pi}} \frac{N}{m} \sqrt{e^{2\pi m^2} - 1} e^{-\pi m^2 \theta(-\sigma)} \cos \left[ \frac{\sigma}{2} - m^2 \log(|\sigma|) + \arg \Gamma(1 + im^2) \right] [1 + \mathcal{O}(\sigma^{-1})],\end{aligned}\quad (3.16)$$

where  $\theta(-\sigma) = 0$  ( $= 1$ ) for  $\sigma > 0$  ( $\sigma < 0$ ). Due to the oscillatory behavior of the wave functions, the magnitude of  $N$  cannot be fixed by a normalization integral. In Sec. IV we show that the normalization of highly excited states may be determined using duality between the contributions of bound states and free fermions to current propagators.

So far we neglected an  $\epsilon(x)$  factor in (3.8) and thus assumed<sup>9</sup>  $x \geq 0$ . When the solutions for  $x < 0$  are defined according to the parity constraint (3.6) the choice of phase indicated in (3.15) implies that

$$\begin{aligned}\Phi_1(-x) &= \eta \Phi_1(x), & \Phi_0(-x) &= -\eta \Phi_0(x), \\ \Phi_1(x=0) &= 0 \quad (\eta = -1), & \Phi_0(x=0) &= 0 \quad (\eta = +1).\end{aligned}\quad (3.17)$$

The latter conditions ensure the continuity at  $x = 0$  of  $\Phi_1(x)$ ,  $\Phi_0(x)$  and their derivatives. They also determine the discrete bound-state masses  $M$  by the positions of the zeros of  $\Phi_0(\sigma)$  and  $\Phi_1(\sigma)$  for  $\eta = \pm 1$ , respectively, through  $\sigma(x=0) = M^2$  according to (3.3).

In Fig. 3(a) we compare the symmetric ( $\eta = 1$ ) ground state mass as a function of the constituent fermion mass  $m$  with the solutions of the Schrödinger equation (2.18) (at the reduced mass  $m/2$ ). As expected there is good agreement for  $m \gtrsim 1$ . The mass difference decreases strictly monotonously with  $m$ , as shown on a logarithmic scale in Fig. 3(b).

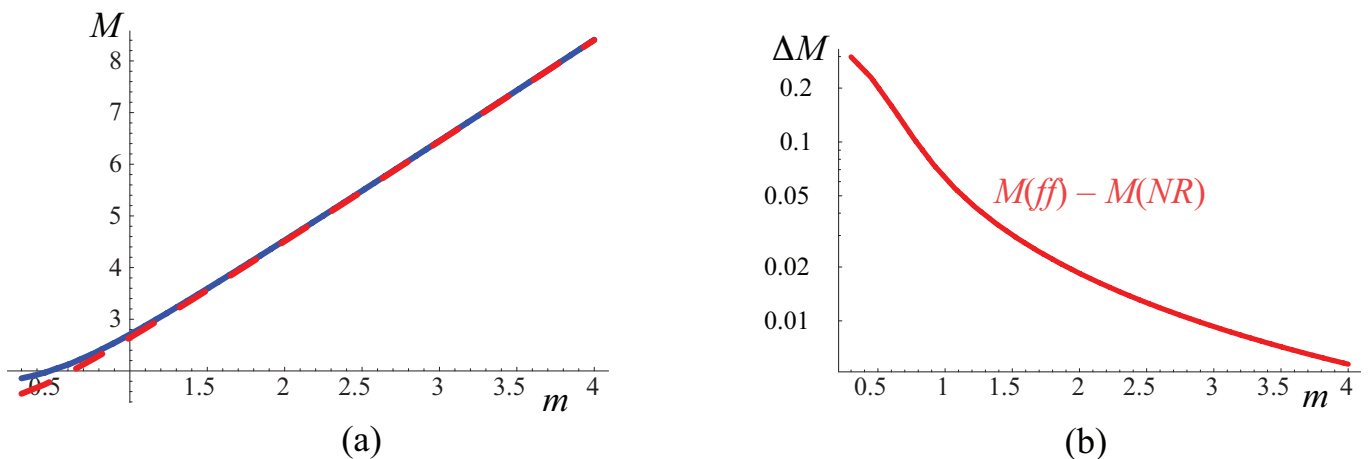


FIG. 3: The ground state mass  $M$  as a function of the fermion mass  $m$  [in units of the coefficient of the linear potential (2.2)]. (a) The solid blue curve is for the  $f\bar{f}$  wave function (3.15). The dashed red curve shows  $M = 2m + E_b$ , with  $E_b$  the eigenvalue given by the Schrödinger equation (2.17) (with reduced mass  $\frac{1}{2}m$ ). (b) The difference of the ground state masses in (a), plotted versus  $m$  on a logarithmic scale.

In Fig. 4 we compare the shape of the ( $P = 0$ ) symmetric  $\Phi_1(x)$  ground state wave function in (3.15) for  $m = 4$  with the corresponding Schrödinger wave function (2.18). The two wave functions are nearly equal at low  $x$  where  $V(x) \ll m$ . The relativistic  $f\bar{f}$  wave function increases from small values in the intermediate  $x$  region to begin its asymptotic oscillations near  $V = 2M$ .  $\Phi_1(x)$  is symmetric in the region  $0 \leq V(x) \leq 2M$  since it depends on  $x$  only via  $\sigma = [M - V(x)]^2$ .

<sup>9</sup> The solutions actually take the form of (3.15) also for  $x < 0$  when  $m_1 = m_2$ , but the extension to negative  $x$  is still non-trivial as the mapping  $\sigma = \sigma(x)$  has a kink at  $x = 0$ .

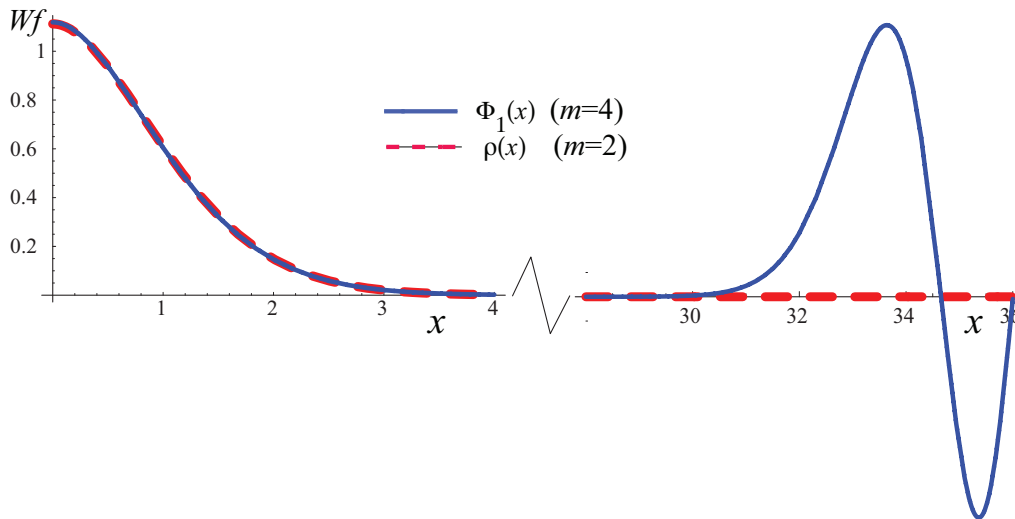


FIG. 4: The ground state  $f\bar{f}$  wave function  $\Phi_1(x)$  in (3.15) for  $P = 0$  and  $m = 4$  (solid blue line) compared to the non-relativistic Schrödinger wave function  $\rho(x)$  in (2.18) with the reduced mass  $m = 2$  (dashed red line). The argument of  $\Phi_1$  is  $\sigma = [M - V(x)]^2$  with  $M = 8.4100$  while the binding energy for  $\rho(x)$  is  $E_b = 0.4043$ . Both wave functions are normalized to unity in the region  $0 \leq x \leq 5$ .

The spectrum of  $f\bar{f}$  states is shown in the form of “Regge trajectories” in Fig. 5(a), where the square of the bound-state masses  $M_n^2$  are shown as a function of the excitation number  $n$ . The trajectories of the symmetric and antisymmetric ( $\eta = \pm 1$ ) solutions are degenerate, and in the case of small constituent mass  $m = 0.1$  almost linear. For  $m = 4$  the trajectory initially has a smaller slope.

Notice that the lowest antisymmetric ( $\eta = -1$ ) state has  $M = 0$  for any  $m$ , since  $\Phi_1(\sigma = 0) = 0$ . This state was not included in Fig. 5(a). Fig. 5(b) shows that the wave function  $\Phi_0(x)$  of this state is essentially non-vanishing only in the relativistic region,  $V(x) \gtrsim 2m$ . In the non-relativistic limit the wave function thus tends to zero.

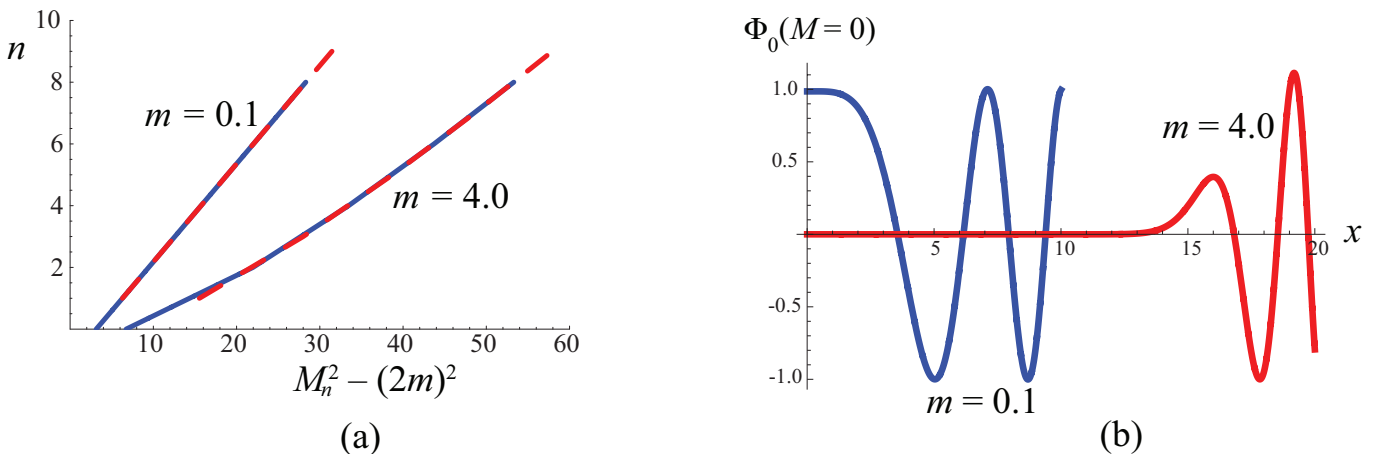


FIG. 5: (a) Regge trajectories for  $f\bar{f}$  bound states with equal constituent masses,  $m = 0.1$  (upper curve) and  $m = 4.0$  (lower curve). The excitation number  $n$  of the first five symmetric ( $\eta = +1$ ,  $n = 0, 2, \dots, 8$ , blue lines) and antisymmetric ( $\eta = -1$ ,  $n = 1, 3, \dots, 9$ , dashed red lines) states are plotted versus  $M_n^2 - (2m)^2$ . (b) The wave function  $\Phi_0(x)$  of the  $\eta = -1$  state with  $M = 0$  [not included in (a)] for  $m = 0.1$  and  $m = 4.0$ . The normalization is chosen arbitrarily such that  $\Phi_0 = -1$  at the first minimum.

The mass spectrum can be solved analytically for high excitations, since then  $\sigma(x = 0) = M^2$  is large at the positions  $x = 0$ , where the wave functions (3.15) must vanish according to (3.17), and their asymptotic expressions

(3.16) may be used. The zeros are thus approximately at

$$\frac{\sigma}{2} - m^2 \log(\sigma) + \arg \Gamma(1 + im^2) = \begin{cases} \pi(n + \frac{1}{2}) & (\eta = +1) \\ \pi n & (\eta = -1) \end{cases}. \quad (3.18)$$

This may be combined into the asymptotic mass spectrum

$$M_n^2 = \pi n + 2m^2 \log(\pi n) - 2 \arg \Gamma(1 + im^2) + \mathcal{O}(n^{-1}), \quad (3.19)$$

where  $n$  is odd (even) for  $\eta = +1$  ( $\eta = -1$ ).

On the other hand, in the limit of small fermion mass,  $m \rightarrow 0$ , we have  $u^{-im^2}(1-u)^{im^2} \simeq 1 + im^2 \log[(1-u)/u]$  in the integral representation (3.15) for  $\Phi_1$ . Hence we find to  $\mathcal{O}(m^2)$ ,

$$\Phi_1(\sigma) = N(1 - \frac{1}{2}\pi m^2) \left[ 2 \sin\left(\frac{\sigma}{2}\right) + im^2 \sigma e^{-i\sigma/2} \int_0^1 du e^{i\sigma u} \log\left(\frac{1-u}{u}\right) + \mathcal{O}(m^4) \right]. \quad (3.20)$$

For  $\sigma > 0$  the integral can be expressed in terms of sine and cosine integral functions as

$$i\sigma e^{-i\sigma/2} \int_0^1 du e^{i\sigma u} \log\left(\frac{1-u}{u}\right) = 2 \cos\left(\frac{\sigma}{2}\right) [\text{Ci}(\sigma) - \log(\sigma) - \gamma_E] + 2 \sin\left(\frac{\sigma}{2}\right) \text{Si}(\sigma), \quad (3.21)$$

where  $\gamma_E = 0.577216$  is Euler's constant, and

$$\text{Si}(z) = \frac{\pi}{2} - \int_z^\infty du \frac{\sin(u)}{u}, \quad \text{Ci}(z) = - \int_z^\infty du \frac{\cos(u)}{u}. \quad (3.22)$$

Imposing the continuity conditions (3.17) at  $x = 0$  we find the spectrum

$$M_n^2 = \pi n + 2m^2 [\log(\pi n) - \text{Ci}(\pi n) + \gamma_E] + \mathcal{O}(m^4); \quad n = 0, 1, 2, \dots, \quad (3.23)$$

where  $n$  is odd (even) for  $\eta = +1$  ( $\eta = -1$ ). The case  $n = 0$  should be understood by taking the limit  $n \rightarrow 0$ , which gives  $M_0^2 = 0$ , the solution shown in Fig. 5(b), which is exact for any  $m$ .

For  $m$  exactly equal to zero the full wave function (3.7) reduces to  $\Phi(\sigma) = -2iN \exp(i\sigma_1\sigma/2)$ , which is regular at all  $\sigma$ . Hence there is no constraint on the spectrum when  $m = 0$ . On the other hand, (3.23) gives  $M_n^2 = \pi n$  in the  $m \rightarrow 0$  limit. The discrete spectrum obtained for regular solutions when  $m \neq 0$  thus differs, even in the  $m \rightarrow 0$  limit, from the continuous spectrum found with  $m = 0$ . Furthermore, the original bound state equation (3.2) implied parity doubling when  $m_1 = m_2 = 0$ : The parity transformed wave function  $\gamma^0 \Phi(-x) \gamma^0$  is a solution (with  $P \rightarrow -P$ ) having the same eigenvalue  $E$  as  $\Phi(x)$ . To the contrary, the  $m \rightarrow 0$  states have parity  $\eta = (-1)^{n+1}$  and are *not* parity degenerate.

## B. $f\bar{f}$ solutions for $m_1 \neq m_2$

The general solution of (3.9) when

$$\Delta m^2 \equiv m_1^2 - m_2^2 \neq 0 \quad (3.24)$$

is

$$\begin{aligned} \Phi_1(\sigma) + \Phi_0(\sigma) &= e^{i\sigma/2} \{ |\sigma|^{-\frac{i}{2}\Delta m^2} (a+ib) m_{1\ 1} F_1(im_2^2, 1-i\Delta m^2, -i\sigma) - |\sigma|^{\frac{i}{2}\Delta m^2} (a-ib) m_{2\ 1} F_1(im_1^2, 1+i\Delta m^2, -i\sigma) \}, \\ \Phi_1(\sigma) - \Phi_0(\sigma) &= e^{-i\sigma/2} \{ |\sigma|^{\frac{i}{2}\Delta m^2} (a-ib) m_{1\ 1} F_1(-im_2^2, 1+i\Delta m^2, i\sigma) - |\sigma|^{-\frac{i}{2}\Delta m^2} (a+ib) m_{2\ 1} F_1(-im_1^2, 1-i\Delta m^2, i\sigma) \}, \end{aligned} \quad (3.25)$$

where  $a$  and  $b$  are complex constants<sup>10</sup>. The parametrization of the constants was chosen such that for real  $a$  and  $b$ ,  $\Phi_1$  is real and  $\Phi_0$  is imaginary as in the case of equal masses above. We may assume that  $x > 0$ , with the solutions of definite parity defined as in (3.6).

<sup>10</sup> The absolute values in  $|\sigma|^{\pm i\Delta m^2/2}$  specify the branch choices for positive and negative  $\sigma$ . This choice is natural in view of our definition of the phase  $\varphi$  in (3.5), which behaves as  $e^{i\varphi} \sim |\sigma|^{\mp i\Delta m^2/2}$  for  $E - V \rightarrow \pm P$ .

The  $2 \times 2$  wave function  $\Phi$  is generally singular at  $\sigma = 0$  according to (3.7), which may be expressed as

$$\Phi = \Phi_0 \left\{ 1 + \frac{m_1 - m_2}{\sigma} [(E - V)\sigma_3 + P i\sigma_2] \right\} + \Phi_1 \left\{ \sigma_1 + \frac{m_1 + m_2}{\sigma} [P\sigma_3 + (E - V)i\sigma_2] \right\}. \quad (3.26)$$

Noting that  ${}_1F_1(\alpha, \beta, 0) = 1$  we find that for  $\sigma = [(E - V) + P][(E - V) - P] \rightarrow 0$ , the most singular terms of the full bound state wave function in (3.1) are

$$e^{i\varphi} \Phi(E - V \rightarrow \pm P) \sim \frac{E - V}{\sigma} \Delta m^2 |\sigma|^{\mp i\Delta m^2} (a \pm ib)(\sigma_3 \pm i\sigma_2). \quad (3.27)$$

No choice of the parameters  $a, b$  can eliminate the  $1/\sigma$  singularity at both  $E - V = P$  and  $E - V = -P$ . The phase  $|\sigma|^{\mp i\Delta m^2}$ , however, ensures the integrability of the wave function when multiplied by any regular function. For example, the orthogonality relations to be discussed below are well-defined.

It is interesting to note that in the infinite momentum frame,  $P \rightarrow +\infty$  (IMF), the full wave function is regular if  $a + ib = 0$ . This choice removes the  $1/\sigma$  singularity in (3.27) at  $E - V = P$ , while in the IMF  $E - V \neq -P$  at finite  $x$ . As  $P \rightarrow \infty$  both square brackets in (3.26) approach  $E(\sigma_3 + i\sigma_2) \equiv E\gamma^+$  and the full  $2 \times 2$  wave function becomes, for  $b = ia$  in (3.25),

$$e^{i\varphi} \Phi_{P \rightarrow \infty}(\sigma) = -2ia M^{i\Delta m^2} \frac{m_1 m_2}{1 + i\Delta m^2} E\gamma^+ e^{-i\sigma/2} {}_1F_1(1 - im_2^2, 2 + i\Delta m^2, i\sigma), \quad (3.28)$$

which is indeed regular at  $\sigma = 0$ , and also square integrable in  $x$ . While  $b = ia$  thus appears to be the most physical choice of parameters, we anyhow continue the discussion assuming generic  $a$  and  $b$ .

By using known identities for the  ${}_1F_1$  functions it is straightforward to check that in the equal-mass limit  $m_1, m_2 \rightarrow m$  the wave function  $\Phi_1(\sigma)$  in (3.25) reduces to

$$\Phi_1(m_1 = m_2 = m) = ibm [e^{i\sigma/2} {}_1F_1(im^2, 1, -i\sigma) - e^{-i\sigma/2} {}_1F_1(-im^2, 1, i\sigma)] = -bm \sigma e^{-i\sigma/2} {}_1F_1(1 - im^2, 2, i\sigma), \quad (3.29)$$

which agrees with our previous expression (3.15) when  $N = -bm$ . The  $m_1 \rightarrow 0$  limit is also simple since  ${}_1F_1(\alpha = 0, \beta, z) = 1$ . Thus

$$\Phi_j(m_1 = 0) = \frac{1}{2} m_2 [- (a - ib) |\sigma|^{-\frac{i}{2} m_2^2} e^{i\sigma/2} + (-1)^j (a + ib) |\sigma|^{\frac{i}{2} m_2^2} e^{-i\sigma/2}] \quad (j = 0, 1). \quad (3.30)$$

The definition (3.6) of  $\Phi(x < 0)$  requires continuity of  $\Phi_0(x = 0)$  and  $\Phi_1(x = 0)$  for the bound-state equation (3.9) to be satisfied at all  $x$ ,

$$\Phi_j^\eta(x = 0) = \eta \Phi_j^{\eta*}(x = 0) \quad (j = 0, 1). \quad (3.31)$$

$$\partial_x \Phi_j^\eta(x = 0) = -\eta \partial_x \Phi_j^{\eta*}(x = 0) \quad (3.32)$$

The condition (3.31) requires that both  $\Phi_0(0)$  and  $\Phi_1(0)$  are real (imaginary) for  $\eta = +1$  ( $\eta = -1$ ). In general, this can be satisfied by adjusting the overall phase in (3.25) provided the phase difference of  $\Phi_0(0)$  and  $\Phi_1(0)$  is 0 or  $\pi$ . This constraint determines the mass spectrum for fixed  $a/b$ . The continuity of the derivatives (3.32) follows from (3.31) and the fact that the wave functions satisfy the bound-state equation (3.9) for  $x > 0$ .

As  $m_1 \rightarrow m_2$  the phase difference between  $\Phi_0$  and  $\Phi_1$  approaches  $\pm\pi/2$  as seen from Eqs. (3.9) and (3.29). Therefore, in the equal-mass case these wave functions cannot have the same phase, instead one of them has to vanish as we found in (3.17). As pointed out above, the same happens if  $a$  and  $b$  are both real (or, more generally, if they have the same phase, *i.e.*,  $a/b$  is real). For  $m_1 \approx m_2$  the wave functions have a relative phase of nearly  $\pm\pi/2$  everywhere except close to a zero of one of them. Thus the spectrum depends smoothly on  $m_1 - m_2$ .

In Fig. 6 we illustrate some properties of the solutions in terms of the density  $|\Phi_0|^2 + |\Phi_1|^2$  for the choice  $b = ia$ . The fermion masses  $m_1 = 1.0$ ,  $m_2 = 1.5$  are fairly high compared to  $V'(x) = 0.5$ , ensuring that the multipair contributions to the wave functions are well separated in the ground state rest frame [solid red curve in (a)]. The wave function of the excited state [dashed blue curve in (a)] extends to larger fermion separations  $x$  before decreasing to small values, but its multipair contributions shift correspondingly in  $x$ , approximately preserving the extent of the gap. The same densities are shown in (b) for the case of non-vanishing center-of-mass momentum,  $P = 5$ . The wave functions Lorentz contract at low  $x$ , whereas the length of the gap to the multiparticle contributions grows.

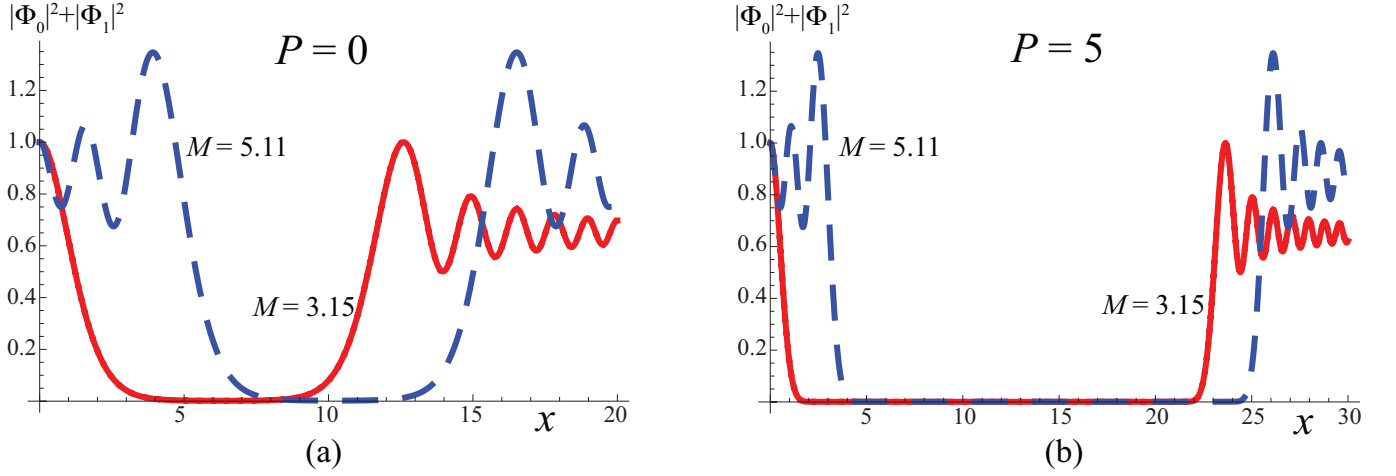


FIG. 6: (a) The density  $|\Phi_0|^2 + |\Phi_1|^2$  as a function of the distance  $x$  between the constituents for the ground state ( $M = 3.15$ , solid red line) and for an excited state ( $M = 5.11$ , dashed blue line). The constituent masses are  $m_1 = 1.0$  and  $m_2 = 1.5$ . (b) The densities in (a) plotted in the case of non-vanishing center-of-mass momentum,  $P = 5.0$ . The densities are symmetric under  $x \rightarrow -x$  and normalized to unity at  $x = 0$ .

### C. Orthogonality of the $f\bar{f}$ states

If we include the phase  $\varphi$  in the definition of the  $2 \times 2$  wave function in the state (3.1),

$$\Psi(x) \equiv e^{i\varphi(x)} \Phi(x), \quad (3.33)$$

the inner product of two states  $k, \ell$  reduces to

$$\langle E_\ell, P_\ell | E_k, P_k \rangle = 2\pi\delta(P_k - P_\ell) \int dx \text{Tr} [\Psi_\ell^\dagger(x) \Psi_k(x)] \quad (3.34)$$

since only the anticommutators of the fields contribute due to (1.2). Thus orthogonality of the states requires

$$\int dx \text{Tr} [\Psi_\ell^\dagger(x) \Psi_k(x)] = 0 \quad (P_k = P_\ell \equiv P, \quad E_k \neq E_\ell). \quad (3.35)$$

The bound-state equations for  $\Psi_k(x)$  and  $\Psi_\ell^\dagger(x)$  are

$$\begin{aligned} i\partial_x \{\sigma_1, \Psi_k\} - \frac{1}{2}P [\sigma_1, \Psi_k] + m_1\sigma_3\Psi_k - m_2\Psi_k\sigma_3 &= [E_k - V]\Psi_k, \\ -i\partial_x \{\sigma_1, \Psi_\ell^\dagger\} + \frac{1}{2}P [\sigma_1, \Psi_\ell^\dagger] + m_1\Psi_\ell^\dagger\sigma_3 - m_2\sigma_3\Psi_\ell^\dagger &= [E_\ell - V]\Psi_\ell^\dagger. \end{aligned} \quad (3.36)$$

Multiplying the first equation by  $\Psi_\ell^\dagger$  from the left, the second by  $\Psi_k$  from the right and then taking the trace of their difference gives

$$i\partial_x \text{Tr} \left( \sigma_1 \left\{ \Psi_\ell^\dagger, \Psi_k \right\} \right) = (E_k - E_\ell) \text{Tr} (\Psi_\ell^\dagger \Psi_k). \quad (3.37)$$

Integrating both sides over all  $x$  the *lhs.* gets a contribution only from  $x = \pm\infty$  [only the  $\Phi_0$  and  $\Phi_1$  components of the  $2 \times 2$  wave function  $\Phi$  in (3.7) contribute on the *lhs.*]. To leading order in the  $x \rightarrow +\infty$  limit we have from (3.5)

and (3.25)

$$\begin{aligned} \varphi(x \rightarrow \infty) &\simeq \Delta m^2 \log[M/(E+P)], \\ \Phi_j(\sigma \rightarrow \infty) &\simeq -\frac{1}{2} [(-1)^j C_1 \sigma^{i(m_1^2+m_2^2)/2} e^{-i\sigma/2} + C_2 \sigma^{-i(m_1^2+m_2^2)/2} e^{i\sigma/2}] \quad (j=0,1), \end{aligned} \quad (3.38)$$

$$C_1 = m_1(a-ib) \frac{\Gamma(1+i\Delta m^2)}{\Gamma(1+im_1^2)} e^{\pi m_2^2/2} - m_2(a+ib) \frac{\Gamma(1-i\Delta m^2)}{\Gamma(1+im_2^2)} e^{\pi m_1^2/2}, \quad (3.39)$$

$$C_2 = m_2(a-ib) \frac{\Gamma(1+i\Delta m^2)}{\Gamma(1-im_2^2)} e^{\pi m_1^2/2} - m_1(a+ib) \frac{\Gamma(1-i\Delta m^2)}{\Gamma(1-im_1^2)} e^{\pi m_2^2/2}. \quad (3.40)$$

The result in the  $x \rightarrow -\infty$  limit is given by the complex conjugate of the above, since  $\Psi(-x) = \eta \Psi^*(x)$  according to (3.6). The product  $\Psi_\ell^\dagger \Psi_k$  in (3.37) oscillates asymptotically, and its integral may be defined analogously to that of plane waves, *e.g.*, by adding a factor  $e^{-\epsilon|x|}$  with infinitesimal  $\epsilon > 0$ . Then the integral of the *lhs.* of (3.37) vanishes, and the orthogonality (3.35) is ensured.

## IV. DUALITY

### A. Wave function normalization

The bound-state equation does not determine the overall normalization (or phase) of the wave functions. Due to contributions from an infinite number of particle pairs the integral of the norm of the Dirac-type wave functions diverges. The relative normalization of high-mass states is needed to determine the parton distributions of the bound states in Sec. VI. Here we shall use an approximate duality relation to determine their normalization. For simplicity we limit ourselves to the equal-mass case in the following,  $m_1 = m_2 = m$ .



FIG. 7: Duality between resonance and fermion loop contributions to the imaginary part of a current propagator. The relation should hold in a semi-local sense, and become more accurate at high excitations.

In our present approximation the bound-state spectrum is a sequence of zero-width resonances. The (properly averaged) bound-state contribution to the imaginary part of a current propagator is expected to equal the contribution of a free fermion loop, as illustrated in Fig. 7. This allows to express the wave function at the origin in terms of a calculable perturbative loop contribution. We get consistent normalizations in all frames for scalar, pseudoscalar, vector and pseudovector currents. Furthermore, in the next subsection we show that the bound-state wave functions of highly excited states agree with those of free fermions also for finite separations provided that the potential is much smaller than the energy,  $V \ll E$ .

Denoting the 2-momentum of the current by  $P = (P^0 > 0, P^1)$  the fermion loop amplitude in Fig. 7 is for a vector current

$$L^{\mu\nu}(P) = i \int d^2z \langle 0 | T[j^\mu(z) j^\nu(0)] | 0 \rangle e^{iP \cdot z} = i \int \frac{d^2k}{(2\pi)^2} \text{Tr} \left[ \frac{\not{k} + m}{k^2 - m^2 + i\epsilon} \gamma^\mu \frac{\not{k} - \not{P} + m}{(k-P)^2 - m^2 + i\epsilon} \gamma^\nu \right]. \quad (4.1)$$

The imaginary parts of the loop contribution for this as well as the scalar, pseudoscalar, and pseudovector currents



are<sup>11</sup>

$$\text{Im}L^{\mu\nu}(P) = \left( -g^{\mu\nu} + \frac{P^\mu P^\nu}{P^2} \right) \frac{2m^2}{\sqrt{P^2(P^2 - 4m^2)}} \quad (\text{vector : } j^\mu = \bar{\psi}\gamma^\mu\psi), \quad (4.2)$$

$$\text{Im}L_S(P) = \frac{\sqrt{P^2 - 4m^2}}{2\sqrt{P^2}} \quad (\text{scalar : } j_S = \bar{\psi}\psi), \quad (4.3)$$

$$\text{Im}L_5(P) = \frac{\sqrt{P^2}}{2\sqrt{P^2 - 4m^2}} \quad (\text{pseudoscalar : } j_5 = \bar{\psi}\gamma_5\psi), \quad (4.4)$$

$$\text{Im}L_5^{\mu\nu}(P) = \frac{P^\mu P^\nu}{P^2} \frac{2m^2}{\sqrt{P^2(P^2 - 4m^2)}} \quad (\text{pseudovector : } j_5^\mu = \bar{\psi}\gamma_5\gamma^\mu\psi). \quad (4.5)$$

The contribution of a bound state  $|n\rangle$ , with  $\hat{P}^\mu |n\rangle = P_n^\mu |n\rangle$ , to the imaginary part of the vector current with  $P^0 > 0$  is

$$\begin{aligned} \text{Im}L_n^{\mu\nu}(P) &= \text{Im} i \int d^2x e^{i(P-P_n)\cdot x} \langle 0|j^\mu(0)|n, t=0\rangle \langle n, t=0|j^\nu(0)|0\rangle \theta(x^0) \\ &= 2\pi^2 \delta^2(P - P_n) \langle 0|j^\mu(0)|n\rangle \langle n|j^\nu(0)|0\rangle. \end{aligned} \quad (4.6)$$

The expressions (3.1) and (3.7) for an equal-mass bound state are

$$|n\rangle = \int dx_1 dx_2 e^{iP_n^1(x_1+x_2)/2} \bar{\psi}(0, x_1) \Phi(x_1 - x_2) \psi(0, x_2) |0\rangle_R, \quad (4.7)$$

$$\Phi(x) = \Phi_0(x) + \Phi_1(x) \gamma_5 \left[ 1 - \frac{2m}{\sigma} \mathbb{I}^\dagger \right], \quad (4.8)$$

where  $\gamma_5 = \gamma^0\gamma^1$ . Using  $\text{Tr}(\gamma_5\gamma^\mu\gamma^\nu) = 2\epsilon^{\mu\nu}$ , where  $\epsilon^{01} = 1$ , we get

$${}_R\langle 0|j^\mu(0)|n\rangle = {}_R\langle 0|\bar{\psi}(0)\gamma^\mu\psi(0)|n\rangle = \text{Tr}[\gamma^\mu\gamma^0\Phi(0)\gamma^0] = -\frac{4m}{P^2}\Phi_1(0)\epsilon^{\mu\nu}P_\nu. \quad (4.9)$$

We note that  ${}_R\langle 0|j^\mu(0)|n\rangle P_\mu = 0$  as required by gauge invariance. The sum over intermediate states  $\sum_n |n\rangle\langle n|$  includes an integral over the momentum  $P_n^1$  of the single bound state considered in (4.6). The (local) average over the energy  $P_n^0$  of this bound-state contribution should be dual to the loop contribution (4.2),

$$\frac{1}{\Delta M_n^2} \int \frac{d^2P_n}{(2\pi)^2} \text{Im}L_n^{\mu\nu}(P) \simeq \text{Im}L^{\mu\nu}(P). \quad (4.10)$$

According to (3.19) the bound-state separation  $\Delta M_n^2 = M_{n+1}^2 - M_n^2 \simeq 2\pi$  at large masses [for states with  $\Phi_1(0) \neq 0$ ]. Eq. (4.10) then gives

$$|\Phi_1(x=0)|^2 \simeq \frac{\pi}{2} \sqrt{\frac{M_n^2}{M_n^2 - 4m^2}} \quad (\text{vector current}). \quad (4.11)$$

Based on the form of the bound-state equation (3.9) we previously noted that  $\Phi_1(\sigma)$  is frame independent. Since  $\sigma(x=0) = M^2$  is also frame independent this implies that  $\Phi_1(x=0)$  cannot depend on  $P_n^1$ . The duality relation (4.11) satisfies this constraint.

We may determine the normalization analogously using currents with other Lorentz structures. Thus

$${}_R\langle 0|\bar{\psi}(0)\psi(0)|n\rangle = \text{Tr}[\gamma^0\Phi(0)\gamma^0] = 2\Phi_0(0) \quad (\text{scalar}), \quad (4.12)$$

$${}_R\langle 0|\bar{\psi}(0)\gamma_5\psi(0)|n\rangle = \text{Tr}[\gamma_5\gamma^0\Phi(0)\gamma^0] = -2\Phi_1(0) \quad (\text{pseudoscalar}), \quad (4.13)$$

$${}_R\langle 0|\bar{\psi}(0)\gamma_5\gamma^\mu\psi(0)|n\rangle = \text{Tr}[\gamma_5\gamma^\mu\gamma^0\Phi(0)\gamma^0] = -\frac{4m}{P^2}\Phi_1(0)P^\mu \quad (\text{pseudovector}). \quad (4.14)$$

<sup>11</sup> In  $D = 1 + 1$  dimensions the pseudovector and vector currents are related,  $\bar{\psi}\gamma_5\gamma^\mu\psi = \epsilon^{\mu\nu}\bar{\psi}\gamma_\nu\psi$ , where  $\gamma_5 = \gamma^0\gamma^1$  and  $\epsilon^{01} = 1$ . For completeness we anyhow give results for both.

The duality relation for the scalar current gives

$$|\Phi_0(0)|^2 \simeq \frac{\pi}{2} \sqrt{\frac{M_n^2 - 4m^2}{M_n^2}} \quad (\text{scalar current}). \quad (4.15)$$

According to (3.17) either  $\Phi_0(0) = 0$  or  $\Phi_1(0) = 0$  for any given bound state, so (4.15) is consistent with and complements the condition (4.11) for  $\Phi_1(0)$ . The duality conditions on  $\Phi_1(0)$  obtained using the pseudoscalar and pseudovector currents agree with the vector current relation (4.11).

For highly excited states we thus find that  $|\Phi_0(0)|^2 \simeq \frac{\pi}{2} [|\Phi_1(0)|^2 \simeq \frac{\pi}{2}]$  if  $\Phi_1(0) = 0$  [ $\Phi_0(0) = 0$ ]. Using the asymptotic formula (3.16) we find that, up to an irrelevant phase,

$$N = \frac{\pi m}{2\sqrt{e^{2\pi m^2} - 1}} \quad (4.16)$$

in (3.15).

### B. Wave function in momentum space at large $M$

It is interesting to consider the duality indicated in Fig. 7 also more differentially: Does the wave function of highly excited bound states resemble the free fermion distribution given by the imaginary part of the loop, as expected in the parton model? For large masses,  $M \gg m$ , we may set  $m = 0$ . In the rest frame ( $P^1 = 0$ ) the free fermion momenta are  $k = \frac{1}{2}M(1, \pm 1)$ .

The bound state (3.1) is in the rest frame

$$|M, 0\rangle = \int dx_1 dx_2 \int \frac{dk_1 dk_2}{(2\pi)^2 4|k_1||k_2|} \left[ \bar{u}(k_1) e^{-ik_1 x_1} b_{k_1}^\dagger + \bar{v}(k_1) e^{ik_1 x_1} d_{k_1} \right] \Phi(x_1 - x_2) \left[ u(k_2) e^{ik_2 x_2} b_{k_2} + v(k_2) e^{-ik_2 x_2} d_{k_2}^\dagger \right] |0\rangle_R. \quad (4.17)$$

The  $m = 0$  spinors

$$u(k) = \frac{|k|\sigma_3 - k i\sigma_2}{\sqrt{|k|}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v(k) = -\frac{|k|\sigma_3 - k i\sigma_2}{\sqrt{|k|}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sigma_1 u(k), \quad (4.18)$$

satisfy

$$\begin{aligned} \bar{u}(k)u(k) &= \bar{u}(k)\sigma_1 u(k) = \bar{v}(k)v(k) = \bar{v}(k)\sigma_1 v(k) = 0, \\ \bar{u}(k)v(-k) &= -\bar{v}(k)u(-k) = -2k, \\ \bar{u}(k)\sigma_1 v(-k) &= -\bar{v}(k)\sigma_1 u(-k) = 2|k|. \end{aligned} \quad (4.19)$$

In terms of the Fourier transform of the wave function,

$$\Phi(k) \equiv \int dx \Phi(x) e^{-ikx} = \int dx [\Phi_0(x) + \sigma_1 \Phi_1(x)] e^{-ikx}, \quad (4.20)$$

the state (4.17) becomes

$$|M, 0\rangle = \int \frac{dk}{2\pi 2|k|} \left\{ [-\varepsilon(k)\Phi_0(k) + \Phi_1(k)] b_k^\dagger d_{-k}^\dagger - [\varepsilon(k)\Phi_0(k) + \Phi_1(k)] d_{-k} b_k \right\} |0\rangle_R, \quad (4.21)$$

where  $\varepsilon(k)$  is the sign function. It appears to contain both positive ( $b_k^\dagger d_{-k}^\dagger$ ) and negative ( $d_{-k} b_k$ ) energy modes. The wave function of the negative energy modes, however, turns out to vanish.

For  $m_1 = m_2 = 0$  and  $P^1 = 0$  the bound-state equation (3.9) is

$$i\partial_x \Phi_0(x) = \frac{1}{2}(M - V)\Phi_1(x), \quad i\partial_x \Phi_1(x) = \frac{1}{2}(M - V)\Phi_0(x). \quad (4.22)$$

The solutions with a continuous derivative at  $x = 0$  are

$$\begin{aligned} \Phi_1(x) + \Phi_0(x) &= \sqrt{\frac{\pi}{2}} \exp \left[ -\frac{i}{2} Mx + \frac{i}{8} x^2 \varepsilon(x) \right], \\ \Phi_1(x) - \Phi_0(x) &= \eta \sqrt{\frac{\pi}{2}} \exp \left[ \frac{i}{2} Mx - \frac{i}{8} x^2 \varepsilon(x) \right], \end{aligned} \quad (4.23)$$

with  $\Phi_j(-x) = (-1)^{j+1} \eta \Phi_j(x)$  as in (3.17). The normalization was determined (up to a phase) by the duality conditions (4.11) and (4.15). In momentum space (4.20) we get

$$\begin{aligned}\Phi_1(k) + \Phi_0(k) &= \sqrt{\frac{\pi}{2}} \int dx \exp \left[ -i\left(\frac{1}{2}M + k\right)x + ix^2 \varepsilon(x)/8 \right] \simeq \frac{1}{2}(2\pi)^{3/2} \delta(k + \frac{1}{2}M), \\ \Phi_1(k) - \Phi_0(k) &\simeq \frac{1}{2}\eta(2\pi)^{3/2} \delta(k - \frac{1}{2}M),\end{aligned}\tag{4.24}$$

where the approximation is valid in the large mass limit,  $M \gg 1$ . Consequently

$$\varepsilon(k)\Phi_0(k) + \Phi_1(k) = 0,\tag{4.25}$$

and the bound state (4.21) reduces to

$$|M, 0\rangle = \frac{\sqrt{2\pi}}{2M} \left( \eta b_{M/2}^\dagger d_{-M/2}^\dagger + b_{-M/2}^\dagger d_{M/2}^\dagger \right) |0\rangle_R,\tag{4.26}$$

with a momentum distribution of free fermions, in agreement with the parton model. The approximation made in (4.24) breaks down at large fermion separations where  $V(x) \gtrsim M$  and the effects of confinement set in. Thus the fermions are approximately free only at shorter distances.

An analogous study of duality may be made in a frame with non-vanishing CM momentum  $P$ . The parton state corresponding to (4.26) is then

$$|E, P\rangle = \frac{\sqrt{2\pi}}{2M} \left( \eta b_{E/2+P/2}^\dagger d_{-E/2+P/2}^\dagger + b_{-E/2+P/2}^\dagger d_{E/2+P/2}^\dagger \right) |0\rangle_R.\tag{4.27}$$

## V. ELECTROMAGNETIC FORM FACTORS

### A. Definition ( $m_1 \neq m_2$ )

The electromagnetic current is

$$j^\mu(z) = \sum_{f=1}^2 e_f \bar{\psi}_f(z) \gamma^\mu \psi_f(z) = e^{i\hat{P}\cdot z} j^\mu(0) e^{-i\hat{P}\cdot z},\tag{5.1}$$

where  $e_f$  is the electric charge of flavor  $f$  and  $\hat{P} = (\hat{P}^0, \hat{P}^1)$  is the generator of time and space translations. We consider the matrix element of  $j^\mu(z)$  between bound states of the form (3.1),

$$|A(P_a)\rangle = \int dx_1 dx_2 \exp \left[ \frac{1}{2} i P_a^1 (x_1 + x_2) \right] \bar{\psi}_1(0, x_1) \Psi_A(x_1 - x_2) \psi_2(0, x_2) |0\rangle_R,\tag{5.2}$$

where we used the notation (3.33) for the  $2 \times 2$  wave function  $\Psi_A$ , whose structure is given by (3.5) and (3.7). Since the bound states are eigenstates of energy and momentum,  $\hat{P}^\mu |A(P_a)\rangle = P_a^\mu |A(P_a)\rangle$ , the form factor can be expressed as

$$F_{AB}^\mu(z) = \langle B(P_b) | j^\mu(z) | A(P_a) \rangle = e^{i(P_b - P_a)\cdot z} \langle B(P_b) | j^\mu(0) | A(P_a) \rangle.\tag{5.3}$$

Due to the retarded boundary conditions,  $\psi_1(z) |0\rangle_R = \psi_2^\dagger(z) |0\rangle_R = 0$ , only anticommutators between the fields of the current with those of the states contribute. In effect, the states  $|A\rangle$  and  $\langle B|$  are treated as the free  $|in\rangle$  and  $\langle out|$  states of standard perturbation theory. Here the asymptotic states are bound by the instantaneous Coulomb potential (1.10) arising from the boundary condition (1.6) on  $A^0$  and have no  $\mathcal{O}(e^2)$  contributions. We expect that a perturbative expansion can be formulated analogously to the one for free asymptotic states. In the following we restrict ourselves to the lowest-order contribution.

Since the potential is confining we consider only neutral bound states, and thus set  $e_1 = e_2 = 1$  in (5.1). Then the

current couples equally to both flavors,

$$\begin{aligned}
F_{AB}^\mu(z) &= \sum_{f=1}^2 F_{AB}^{(f)\mu}(z) = e^{i(P_b - P_a) \cdot z} \int dx_1 dx_2 dy_1 dy_2 e^{i(x_1 + x_2)P_a^1/2 - i(y_1 + y_2)P_b^1/2} \\
&\times {}_R\langle 0 | \psi_2^\dagger(0, y_2) \Psi_B^\dagger(y_1 - y_2) \gamma^0 \psi_1(0, y_1) \sum_{f=1}^2 \bar{\psi}_f(0, 0) \gamma^\mu \psi_f(0, 0) \bar{\psi}_1(0, x_1) \Psi_A(x_1 - x_2) \psi_2(0, x_2) | 0 \rangle_R \\
&= e^{i(P_b - P_a) \cdot z} \int dx e^{i(P_b^1 - P_a^1)x/2} \left\{ \text{Tr} [\Psi_B^\dagger(x) \gamma^\mu \gamma^0 \Psi_A(x)] - \eta_a \eta_b \text{Tr} [\Psi_B(x) \gamma^0 \gamma^\mu \Psi_A^\dagger(x)] \right\}, \tag{5.4}
\end{aligned}$$

where, in the second ( $f = 2$ ) term, we used  $(\gamma^\mu)^T \gamma^0 = \gamma^0 \gamma^\mu$  (since  $\gamma^0 = \sigma_3$ ,  $\gamma^1 = i\sigma_2$ ) and  $\Psi(-x) = \eta \Psi^*(x)$  according to (3.6).

## B. Gauge invariance ( $D = 3 + 1$ )

Gauge invariance of the form factor (5.3) requires that

$$G_{AB}^{(f)}(z) \equiv \partial_\mu^z F_{AB}^{(f)\mu}(z) = 0 \quad (f = 1, 2) \tag{5.5}$$

separately for the current of each flavor  $f$ . We shall show that (5.5) is a consequence of the bound-state equations satisfied by  $\Psi_A$  and  $\Psi_B$ . Since the derivation is essentially independent of the masses and of the number of space-time dimensions we here consider  $m_1 \neq m_2$  and  $D = 3 + 1$ . Then

$$\begin{aligned}
F_{AB}^{(1)\mu}(z) &= e^{i(P_b - P_a) \cdot z} \int d\mathbf{x} e^{i(\mathbf{P}_b - \mathbf{P}_a) \cdot \mathbf{x}/2} \text{Tr} [\Psi_B^\dagger(\mathbf{x}) \gamma^\mu \gamma^0 \Psi_A(\mathbf{x})], \\
G_{AB}^{(1)}(0) &= i \int d\mathbf{x} e^{i(\mathbf{P}_b - \mathbf{P}_a) \cdot \mathbf{x}/2} \text{Tr} [\Psi_B^\dagger(\mathbf{x}) (\not{\mathbf{P}}_B - \not{\mathbf{P}}_A) \gamma^0 \Psi_A(\mathbf{x})]. \tag{5.6}
\end{aligned}$$

Due to translation invariance we set  $z = 0$  in  $G_{AB}(z)$  without loss of generality. The bound-state equations (1.9) for  $\Psi_A(\mathbf{x})$  and  $\Psi_B^\dagger(\mathbf{x})$  are

$$\begin{aligned}
i \nabla_x \cdot \{ \gamma^0 \gamma, \Psi_A \} - \frac{1}{2} \mathbf{P}_a \cdot [ \gamma^0 \gamma, \Psi_A ] + m_1 \gamma^0 \Psi_A - m_2 \Psi_A \gamma^0 &= (E_a - V) \Psi_A, \\
-i \nabla_x \cdot \{ \gamma^0 \gamma, \Psi_B^\dagger \} + \frac{1}{2} \mathbf{P}_b \cdot [ \gamma^0 \gamma, \Psi_B^\dagger ] + m_1 \Psi_B^\dagger \gamma^0 - m_2 \gamma^0 \Psi_B^\dagger &= (E_b - V) \Psi_B^\dagger. \tag{5.7}
\end{aligned}$$

Multiplying the first equation by  $-\Psi_B^\dagger$  from the left, the second by  $\Psi_A$  from the right, and taking the trace of their sum gives

$$-i \nabla_x \cdot \text{Tr} \left( \gamma^0 \gamma \{ \Psi_B^\dagger, \Psi_A \} \right) + \frac{1}{2} (\mathbf{P}_b - \mathbf{P}_a) \cdot \text{Tr} \left( \gamma^0 \gamma [ \Psi_B^\dagger, \Psi_A ] \right) = (E_b - E_a) \text{Tr} \left( \Psi_B^\dagger \Psi_A \right). \tag{5.8}$$

Using  $[ \Psi_B^\dagger, \Psi_A ] = \{ \Psi_B^\dagger, \Psi_A \} - 2 \Psi_A \Psi_B^\dagger$  and multiplying both sides by  $\exp[i(\mathbf{P}_b - \mathbf{P}_a) \cdot \mathbf{x}/2]$  we find

$$-i \nabla_x \cdot \left[ e^{i(\mathbf{P}_b - \mathbf{P}_a) \cdot \mathbf{x}/2} \text{Tr} \left( \gamma^0 \gamma \{ \Psi_B^\dagger, \Psi_A \} \right) \right] = e^{i(\mathbf{P}_b - \mathbf{P}_a) \cdot \mathbf{x}/2} \text{Tr} [\Psi_B^\dagger(\mathbf{x}) (\not{\mathbf{P}}_B - \not{\mathbf{P}}_A) \gamma^0 \Psi_A(\mathbf{x})]. \tag{5.9}$$

Integrating both sides over  $\mathbf{x}$  the *rhs.* becomes  $-i G_{AB}^{(1)}(0)$  and the *lhs.* vanishes (assuming that the integral over the oscillating wave functions is regularized as  $|\mathbf{x}| \rightarrow \infty$ , similarly as for plane waves). This proves the gauge condition (5.5) for  $f = 1$ . For  $f = 2$  the gauge term corresponding to (5.6) is

$$G_{AB}^{(2)}(0) = -i \int d\mathbf{x} e^{-i(\mathbf{P}_b - \mathbf{P}_a) \cdot \mathbf{x}/2} \text{Tr} [\gamma^0 (\not{\mathbf{P}}_B - \not{\mathbf{P}}_A) \Psi_B^\dagger(\mathbf{x}) \Psi_A(\mathbf{x})], \tag{5.10}$$

and the proof that it vanishes is analogous to the above.

### C. Form factor for $m_1 = m_2$ ( $D = 1 + 1$ )

The expression (5.4) for the form factor simplifies in the case of equal masses,  $m_1 = m_2 = m$ . Since  $\varphi = 0$  the structure (3.7) of the wave function  $\Psi = \Phi$  becomes

$$\Phi(x) = \Phi_0(x) + \Phi_1(x)\gamma^0\gamma^1 + 2m\Phi_1(x)\frac{\not{P}^\dagger}{\sigma}\gamma^0\gamma^1, \quad (5.11)$$

where  $\Pi(x) = (P^0 - V(x), P^1)$  is the kinematical 2-momentum (3.4). The traces in (5.4) are now

$$\begin{aligned} \frac{1}{2}\text{Tr} [\Phi_B^\dagger \Phi_A] &= \Phi_{0B}^* \Phi_{0A} + \Phi_{1B}^* \Phi_{1A} \left[ 1 + \frac{4m^2}{\sigma_a \sigma_b} \tilde{\Pi}_a \cdot \Pi_b \right], \\ -\frac{1}{2}\text{Tr} [\Phi_B^\dagger \gamma^1 \gamma^0 \Phi_A] &= \Phi_{0B}^* \Phi_{1A} + \Phi_{1B}^* \Phi_{0A} + \Phi_{1B}^* \Phi_{1A} \frac{4m^2}{\sigma_a \sigma_b} \varepsilon_{\mu\nu} \tilde{\Pi}_a^\mu \Pi_b^\nu, \end{aligned} \quad (5.12)$$

where  $\tilde{\Pi}_a = (P_a^0 - V(x), -P_a^1)$  and  $\varepsilon_{01} = -1$ .

The constraint (5.5) of gauge invariance implies that the form factor in  $D = 1 + 1$  can be expressed as

$$F_{AB}^\mu(q) \equiv \int d^2z F_{AB}^\mu(z) e^{-iq \cdot z} = (2\pi)^2 \delta^2(P_b - P_a - q) \varepsilon^{\mu\nu} q_\nu F_{AB}(Q^2), \quad (5.13)$$

where  $Q^2 = -q^2$ . Solving this for  $F_{AB}(Q^2)$  with  $\mu = 0$ , using Eq. (5.4) for the *lhs.*, and inserting the traces of (5.12), we obtain

$$F_{AB}(Q^2) = -4i \frac{1 - \eta_a \eta_b}{q^1} \int_0^\infty dx \sin\left(\frac{q^1 x}{2}\right) \left[ \Phi_{0B}^*(x) \Phi_{0A}(x) + \Phi_{1B}^*(x) \Phi_{1A}(x) \left( 1 + \frac{4m^2}{\sigma_a \sigma_b} \tilde{\Pi}_a \cdot \Pi_b \right) \right]. \quad (5.14)$$

The form factor vanishes unless  $\eta_a \eta_b = -1$ . Therefore the factor in the square brackets could be taken to be antisymmetric in  $x$ , which allowed us to restrict the integration to positive  $x$ .

## VI. DIS AND PARTON DISTRIBUTIONS

We consider the cross section for  $e(k_1) + A(P_a) \rightarrow e(k_2) + B(P_b)$  in the limit where  $x_{Bj} = Q^2/(2P_a \cdot q)$  is fixed, with  $q = k_1 - k_2$  and  $Q^2 = -q^2$ . The “inclusive” system is thus a discrete bound state  $B$ . The cross section is proportional to the square of the form factor  $F_{AB}(Q^2)$  defined in (5.13), with  $M_b \propto Q$ .

There are some peculiarities with DIS in  $D = 1 + 1$  as compared to  $D = 3 + 1$ :

- The very concept of a “cross section” is related to transverse size. We may nevertheless define a Lorentz-invariant cross section by analogy to the usual case and then compare parton and bound state cross sections.
- The virtual photon has a finite longitudinal (“Ioffe”) coherence length in the target rest frame,  $L_I \simeq Q^{-1} \nu/Q = 1/(2m_a x_{Bj})$ . In the absence of transverse dimensions DIS photons can be coherent on several partons at leading twist. The fractional momentum of a struck parton is kinematically constrained to be  $x_{Bj}$ .
- In  $D = 1 + 1$  the scattering angle can be  $\theta = 0$  (forward) or  $\theta = \pi$  (backward scattering). At the parton-level, forward (elastic) scattering implies  $Q^2 = 0$  and thus is irrelevant for the Bjorken (Bj) limit. In backward scattering  $\hat{s} \simeq -\hat{t} = Q^2$ , which is analogous to standard DIS in the Breit (or brick-wall) frame.
- Since the coupling  $e$  has the dimension of energy in  $D = 1 + 1$ , the parton-level cross section will on dimensional grounds be suppressed (compared to  $D = 3 + 1$ ) by a factor  $e^4/Q^4$ .
- The backward scattering amplitude of elementary spin- $\frac{1}{2}$  fermions is proportional to the fermion masses. This further suppresses their scattering cross section.

### A. The parton distribution

The kinematics of DIS in  $D = 1 + 1$  is discussed in Appendix A. We find that the parton distribution may be expressed as

$$f(x_{Bj}) = \frac{1}{8\pi m^2} \frac{1}{x_{Bj}} |Q^2 F_{AB}(Q^2)|^2. \quad (6.1)$$

where  $m$  is the mass of the target parton and the invariant form factor  $F_{AB}(Q^2)$  is given in (5.14) (for a neutral  $f\bar{f}$  state with  $m_1 = m_2 = m$ ). The mass of the inclusive system in the Bj limit is

$$M_b^2 = Q^2 \left( \frac{1}{x_{Bj}} - 1 \right). \quad (6.2)$$

In order to calculate the leading twist parton distribution for a neutral two-body state we analyze the expression (5.14) at large  $Q^2$  and  $M_b$ . We work in the Breit frame where  $q^0 = P_b^0 - P_a^0 = 0$  and  $q^1 = -Q$  is large. The basic expectation is that the Fourier phase in (5.14) limits the integration to  $x \lesssim 1/Q$ , which is the Lorentz contracted equivalent of a finite (Ioffe) distance in the target rest frame. The integrand is  $\mathcal{O}(Q^0)$  in this region, and the measure adds a factor of  $1/Q$ , so we obtain a contribution at leading order  $F_{AB}(Q^2) \sim 1/Q^2$ . Leading contributions can, however, arise also for larger (typically  $\sim Q^0$ ) values of  $x$ , if the oscillations of the wave functions should cancel the Fourier phase such that a stationary phase arises. Such contributions are analyzed in detail in Appendix B and shown not to affect the leading result.

It is convenient to introduce a rescaled variable

$$v = \frac{xQ}{2}. \quad (6.3)$$

In the Bj limit, taking  $v = \mathcal{O}(Q^0)$  and using the expressions (A.18) for the momenta, the variable  $\sigma$  defined in (3.3) is of  $\mathcal{O}(Q^0)$  for the target,

$$\sigma_a = M_a^2 - \frac{Q}{2x_{Bj}}|x| + \frac{1}{4}x^2 \simeq M_a^2 - \frac{|v|}{x_{Bj}} \equiv \tau_a, \quad (6.4)$$

while it is of  $\mathcal{O}(Q^2)$  for the final state,

$$\sigma_b = Q^2 \left( \frac{1}{x_{Bj}} - 1 \right) - \frac{Q}{2x_{Bj}}|x| + \frac{1}{4}x^2 \simeq Q^2 \left( \frac{1}{x_{Bj}} - 1 \right) - \frac{|v|}{x_{Bj}} \equiv \tau_b. \quad (6.5)$$

Thus we may use the asymptotic forms (3.16) for  $\Phi_B(\sigma_b)$ .

- $\eta_b = -1$

The condition  $\partial_v \Phi_{0B}(v=0) = 0$  of (3.17) determines  $M_b$  such that  $\cos[\frac{1}{2}\tau_b - m^2 \log(\tau_b) + \arg \Gamma(1 + im^2)] = 1$  at  $v=0$  (up to an irrelevant phase). The  $v$  dependence of the logarithm is of  $\mathcal{O}(Q^{-2})$  and may be ignored. As the state is highly excited, we may use the normalization from (4.16). Thus

$$\begin{aligned} \Phi_{0B}(\tau_b) &\simeq -\frac{i\sqrt{2\pi}}{2} \cos\left(\frac{v}{2x_{Bj}}\right), \\ \Phi_{1B}(\tau_b) &\simeq -\frac{\sqrt{2\pi}}{2} \sin\left(\frac{|v|}{2x_{Bj}}\right). \end{aligned} \quad (6.6)$$

The expression (5.14) for the DIS form factor becomes

$$Q^2 F_{AB}(\eta_b = -) \simeq -4i\sqrt{2\pi}(1+\eta_a) \int_0^\infty dv \sin v \left[ \cos\left(\frac{v}{2x_{Bj}}\right) i\Phi_{0A}(\tau_a) - \sin\left(\frac{v}{2x_{Bj}}\right) \Phi_{1A}(\tau_a) \left(1 + \frac{2m^2}{x_{Bj}\tau_a}\right) \right]. \quad (6.7)$$

For large  $v$ ,  $\tau_a \rightarrow -\infty$  and we may use<sup>12</sup> the asymptotic expressions (3.16) also for  $\Phi_A$ :

$$\begin{aligned}\Phi_{0A}(\tau_a) &\simeq -i\sqrt{\frac{2}{\pi}}\frac{N}{m}\sqrt{e^{2\pi m^2}-1}e^{-\pi m^2}\cos\left(\mu(v)-\frac{v}{2x_{Bj}}\right), \\ \Phi_{1A}(\tau_a) &\simeq \sqrt{\frac{2}{\pi}}\frac{N}{m}\sqrt{e^{2\pi m^2}-1}e^{-\pi m^2}\sin\left(\mu(v)-\frac{v}{2x_{Bj}}\right),\end{aligned}\quad (v \rightarrow \infty) \quad (6.8)$$

where

$$\mu(v) = -m^2 \log \frac{v}{x_{Bj}} + \frac{M^2}{2} + \arg \Gamma(1 + im^2), \quad (6.9)$$

and  $M = M_a$  is the target mass. The term in the square brackets in (6.7) behaves as

$$\cos\left(\frac{v}{2x_{Bj}}\right)i\Phi_{0A}(\tau_a) - \sin\left(\frac{v}{2x_{Bj}}\right)\Phi_{1A}(\tau_a)\left(1 + \frac{2m^2}{x_{Bj}\tau_a}\right) \simeq \sqrt{\frac{2}{\pi}}\frac{N}{m}\sqrt{e^{2\pi m^2}-1}e^{-\pi m^2}\cos\mu(v) \quad (6.10)$$

for  $v \rightarrow \infty$ . Because the integrand of (6.7) oscillates asymptotically, the integral does not converge. It can, however, be defined in the standard fashion by adding a “convergence factor”, *e.g.*, a factor  $e^{-\epsilon v}$  in the integrand, and by taking  $\epsilon \rightarrow 0$  in the end.

Recall that we assumed that  $F_{AB}(Q^2)$  only receives leading contributions from the region  $x \sim 1/Q$  in order to derive the result (6.7). The fact that the regularization procedure works suggests that this assumption was correct. We analyze this in detail in Appendix B.

- $\eta_b = 1$

The condition  $\Phi_{0B}(v=0) = 0$  of (3.17) determines  $M_b$  such that  $\sin[\frac{1}{2}\tau_b - m^2 \log(\tau_b) + \arg \Gamma(1 + im^2)] = 1$  at  $v = 0$ . Hence,

$$\begin{aligned}\Phi_{0B}(\tau_b) &\simeq -\frac{i\sqrt{2\pi}}{2}\sin\left(\frac{|v|}{2x_{Bj}}\right), \\ \Phi_{1B}(\tau_b) &\simeq \frac{\sqrt{2\pi}}{2}\cos\left(\frac{v}{2x_{Bj}}\right).\end{aligned} \quad (6.11)$$

The expression (5.14) of the DIS form factor becomes

$$Q^2 F_{AB}(\eta_b = +) \simeq -4i\sqrt{2\pi}(1 - \eta_a) \int_0^\infty dv \sin v \left[ \sin\left(\frac{v}{2x_{Bj}}\right)i\Phi_{0A}(\tau_a) + \cos\left(\frac{v}{2x_{Bj}}\right)\Phi_{1A}(\tau_a)\left(1 + \frac{2m^2}{x_{Bj}\tau_a}\right) \right]. \quad (6.12)$$

## B. Numerical evaluation of the parton distribution

Let us insert the explicit results (3.15) in the expression (6.7) and evaluate the integral numerically. We limit ourselves to the case  $\eta_a = -\eta_b = +1$ . The case of opposite parities can be analyzed similarly. Since the integral does not converge we need to subtract the divergent term and treat it separately. We write

$$Q^2 F_{AB}(\eta_a = +) = I_1 + I_2, \quad (6.13)$$

---

<sup>12</sup> Recall that we already took  $Q \rightarrow \infty$ , thus more precisely, the asymptotic expressions hold for  $Q \gg v \gg 1$ .

where

$$I_1 = -8i\sqrt{2\pi} \int_0^\infty dv \sin v \left[ \cos\left(\frac{v}{2x_{Bj}}\right) i\Phi_{0A}(\tau_a) - \sin\left(\frac{v}{2x_{Bj}}\right) \Phi_{1A}(\tau_a) \left(1 + \frac{2m^2}{x_{Bj}\tau_a}\right) - \sqrt{\frac{2}{\pi}} \frac{N}{m} \sqrt{e^{2\pi m^2} - 1} e^{-\pi m^2} \cos \mu(v) \right], \quad (6.14)$$

$$I_2 = -\frac{16iN}{m} \sqrt{e^{2\pi m^2} - 1} e^{-\pi m^2} \int_0^\infty dv \sin v \cos \mu(v). \quad (6.15)$$

The divergence now only appears in the second integral, which may be calculated analytically. The standard regularization yields

$$I_2 = -16i \frac{N}{m} \sqrt{e^{2\pi m^2} - 1} e^{-\pi m^2} \cosh\left(\frac{\pi m^2}{2}\right) \cos\left(\frac{M^2}{2} + m^2 \log x_{Bj}\right) |\Gamma(1 + im^2)|. \quad (6.16)$$

The first integral  $I_1$  can be calculated numerically<sup>13</sup>. The parton distribution can then be found by using the formula (6.1).

We show numerical results for the parton distribution in Fig. 8. The target wave function was chosen to be the one with lowest nonzero mass, which indeed has  $\eta_a = +1$ . We used Eq. (4.16) for the normalization of the target wave function, and chose two reference values  $m = 0.1$  and  $m = 1.0$  for the fermion masses. For  $m = 0.1$  the target is highly relativistic, such that  $M \simeq 1.78 \gg 2m$ , whereas for the second choice  $m = 1$  we find that  $M \simeq 2.70$ , *i.e.*, a binding energy smaller than the constituent masses. This is reflected in the low- $x_{Bj}$  behavior of  $f(x_{Bj})$ , which is qualitatively different in the two cases. The red curves show the  $x_{Bj} \rightarrow 0$  expansion of  $f(x_{Bj})$ , which is calculated in Appendix C.

One can check numerically that the integral  $I_2$ , which arises from asymptotic oscillations of the wave functions, dominates the result for small  $x_{Bj}$  and  $m$ . Therefore approximately

$$x_{Bj} f(x_{Bj}) \sim \cos^2(M^2/2 + m^2 \log x_{Bj}) \quad (6.17)$$

in this region. Inserting the result for the bound state mass at small  $m$  with  $n = 1$  from (3.23) we find

$$x_{Bj} f(x_{Bj}) \sim \sin^2[m^2 (\log x_{Bj} + \log \pi - \text{Ci}(\pi) + \gamma_E)] \simeq m^4 (\log x_{Bj} + 1.648)^2. \quad (6.18)$$

Thus the contribution from the oscillations has a node at<sup>14</sup> relatively small  $x_{Bj} \simeq 0.19$  when  $m$  is small, and grows logarithmically  $\sim (\log x_{Bj})^2$  as  $x_{Bj} \rightarrow 0$ , until the distribution “saturates” at  $x_{Bj} \sim e^{-1/m^2}$ .

## VII. CONCLUDING REMARKS

Analytical studies of bound state dynamics are usually based on summing Feynman diagrams, *e.g.*, using Dyson-Schwinger techniques [15, 16]. In the weak coupling limit the dynamics is non-relativistic and determined by the Schrödinger equation. Relativistic, confined states like hadrons may then emerge only when the coupling is strong. In  $D = 1 + 1$  dimensions some all-orders, exact results have been obtained, notably for QED at zero fermion mass (the Schwinger model [17]) and for QCD in the limit of a large number of colors,  $N_c \rightarrow \infty$  (the 't Hooft model [18]). Their study have led to valuable insights – see, *e.g.*, [19–21] and references therein. Bosonisation of the massive Schwinger model furthermore allowed to obtain some approximate results in the strong coupling limit [22]. No analogous results have been found in  $D = 3 + 1$  dimensions. Approximations based on a truncation of the Dyson-Schwinger equations have allowed analytical studies of hadrons in QCD [15, 16], which complement numerical results using lattice methods. Currently holographic approaches to QCD motivated by the AdS/CFT correspondence [23, 24] are under intense study as a means of obtaining results in the strong-coupling limit of the theory.

<sup>13</sup> This integral is still not absolutely convergent, which makes the numerical integration rather tricky. We found the best results by using the ExtrapolatingOscillatory method of NIntegrate in Mathematica. One can also introduce a cutoff at large  $v$  and use standard algorithms for the numerical integration.

<sup>14</sup> The approximation for the location of the node is poor since terms suppressed by  $x_{Bj}$  were neglected.



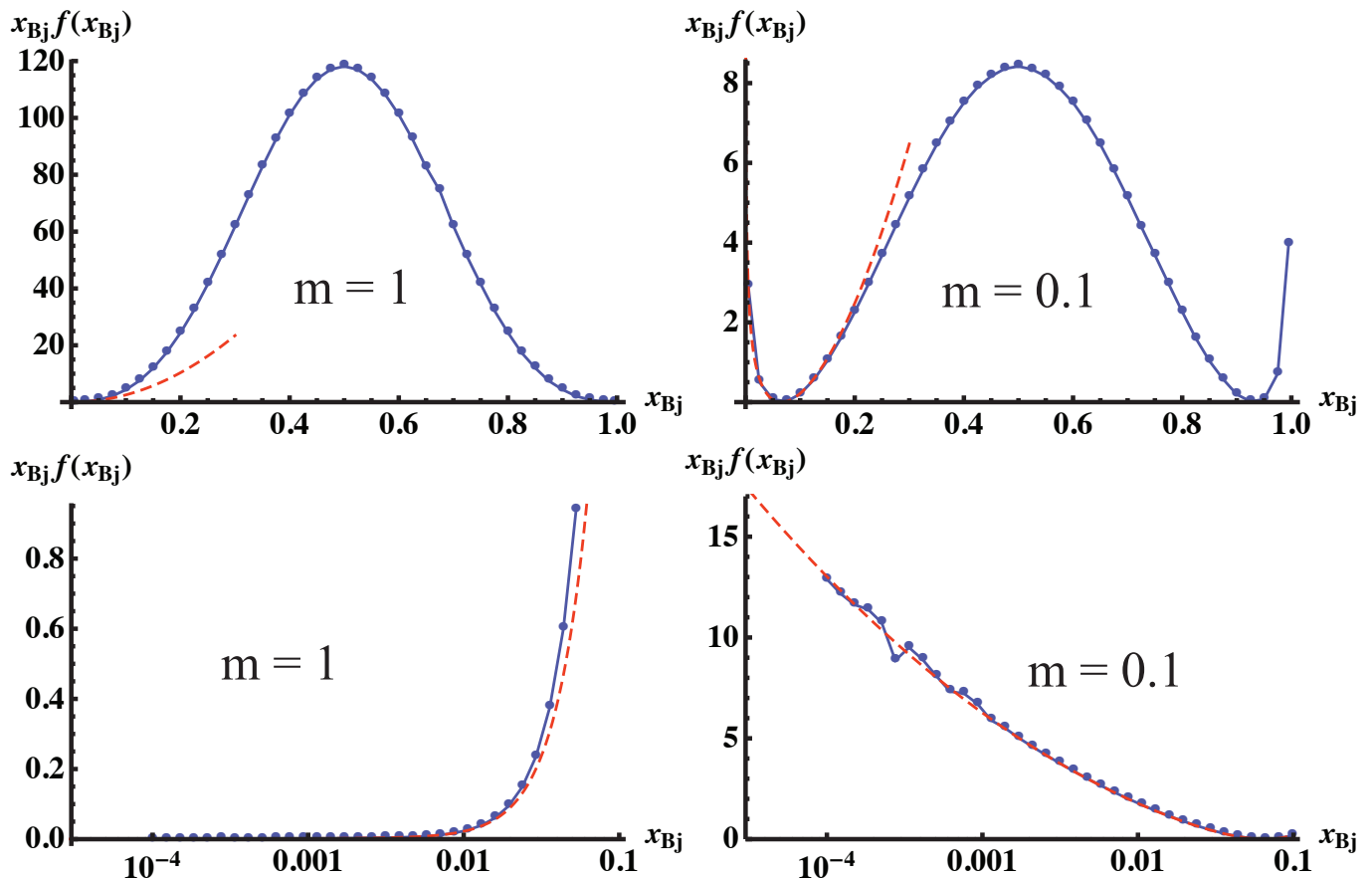


FIG. 8: The parton distribution of the target ground state. We plot the result for  $x_{Bj}f(x_{Bj})$  as a function of  $x_{Bj}$  in linear scale (top) and in logarithmic scale for low  $x_{Bj}$  (bottom). We used  $m = 1$  (left) and  $m = 0.1$  (right) for the fermion masses. The dashed red curves show the asymptotic behavior of  $f(x_{Bj})$  up to next-to-leading order as  $x_{Bj} \rightarrow 0$  [see Eq. (C.14)].

In this paper we explored a rather different approach to relativistic bound states in gauge field theory. It may be relevant provided the QCD coupling remains perturbative even in the long-distance regime governing hadron binding. Our approach is motivated by features of the data discussed in the Introduction, as well as by phenomenological and theoretical studies which indicate that  $\alpha_s$  freezes at a moderate value in the infrared [25–30]. This possibility merits attention since it allows to bring the powerful techniques of perturbation theory to bear on bound state dynamics.

In our scenario the confining potential arises from a boundary condition imposed on the solution of Gauss’ law [5]. This gives an exactly linear  $A^0$  potential even in  $D = 3 + 1$  dimensions, with strength determined by a parameter  $\Lambda$  related to the boundary condition. Since  $\Lambda$  is independent of  $\alpha_s$  the potential is of leading order compared to the perturbative  $\mathcal{O}(\alpha_s)$  interactions. Hence the bound state wave functions have no contributions from transverse gauge bosons at lowest order.

Quark-antiquark pairs will be created even in the lowest order  $A^0$  potential, leading to hadron Fock states with any number of fermions. This is also true for the Dirac states of an electron in an external potential. The Dirac wave function, which has a single electron degree of freedom, is obtained using retarded boundary conditions for the electron propagation. Hence the norm of the Dirac wave function gives the *inclusive* electron density. In our framework mesons are analogously (at lowest order in  $\alpha_s$ ) described by ostensibly two-particle ( $q\bar{q}$ ) wave functions, which implicitly include such quark-antiquark pair production effects. The use of retarded boundary conditions for the quarks is allowed in the external legs of  $S$ -matrix elements since they are amputated and put on-shell. In analogy to the standard perturbative expansion of scattering amplitudes based on the free *in* and *out* states of elementary fields we envisage an expansion based on the Born-level bound states formed by the linear potential.

Previously [7] we verified that the  $f\bar{f}$  Born states are Poincaré covariant in  $D = 1 + 1$ , as expected because the linear potential arises from a boundary condition which is compatible with the field equations of motion. The wave functions turned out to depend on the separation  $x$  between the fermions through the frame-invariant square  $\Pi^2 = \sigma$  of the “kinematical momentum”  $\Pi = (E - V, P)$ , where  $(E, P)$  is the 2-momentum of the bound state and  $V(x)$  is

the linear potential. An earlier study [8] indicated that Poincaré invariance is preserved also in  $D = 3 + 1$ .

In this paper we found the analytic expressions of the Born level  $f\bar{f}$  wave functions in  $D = 1 + 1$  and studied their properties. Their norm tends to a constant at large  $x$ . The nearly non-relativistic case shown in Fig. 4 makes it clear that the constant norm reflects fermion pair production at large values of the potential. The norm of the Dirac wave function behaves similarly (Fig. 2), which in that case implies a continuous Dirac mass spectrum<sup>15</sup> [10–13]. The  $f\bar{f}$  wave function is, however, generally singular at the value of  $x$  where the “kinematical mass” vanishes,  $\Pi^2 = 0$ . Requiring the wave function to be regular gives a discrete spectrum.

We found that the wave functions of highly excited bound states agree with parton model expectations in the range of  $x$  where the potential is small compared to the bound state mass: Only fermions and antifermions of positive energy contribute to the bound state, having the momenta of nearly free particles. Duality allows to determine the overall normalization of the wave functions through the condition that the (average) contribution of the bound states to the imaginary part of current propagators agrees with that of free fermions. The duality relation works in all frames and for all currents.

A manifestation of the virtual fermion pairs in the bound states is provided by the parton distributions measured by deep inelastic lepton scattering. For relativistic states the parton distribution grows as  $x_{Bj} \rightarrow 0$ , qualitatively in agreement with data on sea quarks (although the present calculation is in one space dimension only). The use of retarded boundary conditions allows a simple “valence” description of bound states, while preserving its multiparticle features.

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### Appendix A: DIS in $D = 1 + 1$

#### 1. Units and kinematics in $D = 1 + 1$

The dimension of the Lagrangian is  $[\mathcal{L}] = E^2$  in energy units. Hence the fermion and photon fields have  $[\psi] = E^{1/2}$  and  $[A^\mu] = E^0$ , while the electron charge<sup>16</sup>  $[e] = E^1$ . With standard spinor normalizations  $\bar{u}u = -\bar{v}v = 2m$  the fermion operators have  $[b] = [d] = E^0$  and satisfy  $\{b(k^1), b^\dagger(p^1)\} = 2E_p 2\pi\delta(k^1 - p^1)$ . The states have the same dimensions as their creation operators,  $[|e^-\rangle] = E^0$ . For our bound-state definition (5.2) this implies  $[\Phi] = E^1$ . Consequently the invariant form factor defined in (5.13) is dimensionless,  $[F_{AB}(Q^2)] = E^0$ .

The  $2 \rightarrow 2$  scattering amplitudes

$$\mathcal{A}(e\mu \rightarrow e\mu) = \langle e\mu | T | e\mu \rangle = \mathcal{M}(e\mu \rightarrow e\mu) (2\pi)^2 \delta^2(P_i - P_f) \quad (\text{A.1})$$

have  $[\mathcal{A}] = E^0$  and  $[\mathcal{M}] = E^2$ . Defining the  $2 \rightarrow 2$  “cross section” in analogy to  $D = 3 + 1$  as

$$\sigma_{\text{scat}}(e\mu \rightarrow e\mu) = \frac{1}{2s} \int \frac{dp_1^1 dp_2^1}{(2\pi)^2 4E_1 E_2} |\mathcal{M}|^2 (2\pi)^2 \delta^2(p_1 + p_2 - P_i) \quad (\text{A.2})$$

<sup>15</sup> The Dirac spectrum is continuous for almost all potentials in both two and four dimensions. Curiously, textbooks rarely mention this fact, even though the exceptional case of the  $1/r$  potential in  $D = 3 + 1$  is treated in detail.

<sup>16</sup> For clarity we display the charge  $e$  explicitly in this Appendix, taking  $V(x) = \frac{1}{2}e^2|x|$ .

gives  $[\sigma_{\text{scat}}] = E^0$  as expected. The phase space factor is

$$\sigma_{\text{scat}} = \frac{1}{2s} \int \frac{dp_1^1}{4E_1 E_2} \delta(E_1 + E_2 - P_i^0) |\mathcal{M}|^2 = \frac{1}{2s} \frac{|\mathcal{M}|^2}{4\epsilon_{\mu\nu} p_1^\mu p_2^\nu}, \quad (\text{A.3})$$

where  $\epsilon^{01} = 1$ , and we used

$$\frac{d(E_1 + E_2)}{dp_1^1} = \frac{p_1^1}{E_1} + \frac{p_1^1 - P_i^1}{E_2} = \frac{1}{E_1 E_2} [p_1^1 P_i^0 - p_1^0 P_i^1] = \frac{\epsilon_{\mu\nu} p_1^\mu p_2^\nu}{E_1 E_2}. \quad (\text{A.4})$$

The invariant kinematic factor may be evaluated in the Breit frame where, for  $Q \gg m_1, m_2$ ,

$$p_1 = \frac{Q}{2}(1, 1), \quad p_2 = \frac{Q}{2}(1, -1), \quad (\text{A.5})$$

giving

$$\sigma_{\text{scat}} \simeq \frac{|\mathcal{M}|^2}{4Q^4} \quad \text{for } Q^2 \rightarrow \infty \quad \text{with fixed } m_1, m_2. \quad (\text{A.6})$$

We may evaluate the electron vertex as

$$\begin{aligned} V_e^\mu &= \bar{u}(k_2) \gamma^\mu u(k_1) = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{k_2 + m}{\sqrt{E_2 + m}} \gamma^\mu \frac{k_1 + m}{\sqrt{E_1 + m}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{(E_1 + m)(E_2 + m)}} \left[ (E_1 + m)k_2^\mu + (E_2 + m)k_1^\mu + \frac{1}{2}g^{\mu 0}(k_1 - k_2)^2 \right] \equiv \epsilon^{\mu\nu} q_\nu V(Q), \end{aligned} \quad (\text{A.7})$$

where  $q = k_1 - k_2$ . For backward scattering, taking the incoming electron momentum  $k_1^1 < 0$  and thus  $k_2^1 > 0$ , we may evaluate  $V_e^1$  explicitly as

$$V_e^1 = \sqrt{(E_1 + m)(E_2 - m)} - \sqrt{(E_1 - m)(E_2 + m)} = \frac{(E_1 + m)(E_2 - m) - (E_1 - m)(E_2 + m)}{\sqrt{(E_1 + m)(E_2 - m)} + \sqrt{(E_1 - m)(E_2 + m)}} = -2m \frac{q^0}{Q}, \quad (\text{A.8})$$

which implies

$$V(Q) = \frac{2m}{Q} \quad (\text{A.9})$$

for the invariant part of the vertex in (A.7). Using

$$\epsilon^{\mu\rho} q_\rho \epsilon^{\nu\sigma} q_\sigma = Q^2 \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \quad (\text{A.10})$$

the backward scattering amplitude for  $e\mu \rightarrow e\mu$  is

$$\mathcal{M}(e\mu \rightarrow e\mu) = -e^2 V_e(Q) V_\mu(Q) = -4m_e m_\mu \frac{e^2}{Q^2}. \quad (\text{A.11})$$

Using this in the expression (A.6) for the cross section at large  $Q^2$  gives

$$\sigma_{\text{scat}}(e\mu \rightarrow e\mu) = 4m_e^2 m_\mu^2 \frac{e^4}{Q^8}, \quad (\text{A.12})$$

which is suppressed by  $m_e^2 m_\mu^2 / Q^4$  compared to the “scaling” behavior  $\sim e^4 / Q^4$ .

We may compare the fermion cross section with that for elementary scalars, which do not have a mass suppression. The scalar vertex is

$$V_s^\mu = (k_1 + k_2)^\mu \equiv \epsilon^{\mu\nu} q_\nu V_s(Q). \quad (\text{A.13})$$

We can determine  $V_s(Q)$  from the space component (taking  $k_1^1 < 0$ )

$$V_s^1 = \sqrt{E_2^2 - m^2} - \sqrt{E_1^2 - m^2} = \frac{E_2^2 - E_1^2}{k_2^1 - k_1^1} = q^0 \frac{E_1 + E_2}{q^1} = -q^0 V_s(Q) \quad (\text{A.14})$$

implying

$$V_s(Q) = -\frac{E_1 + E_2}{k_1^1 - k_2^1} = \sqrt{1 + \frac{4m^2}{Q^2}}. \quad (\text{A.15})$$

Consequently the scalar  $2 \rightarrow 2$  amplitude analogous to (A.11) is of  $\mathcal{O}(Q^0)$ , giving a scaling cross section in (A.6),  $\sigma_{\text{scat}} \propto e^4/Q^4$ .

## 2. $e(k_1)A(P_a) \rightarrow e(k_2)B(P_b)$ in the Bj limit

The Bj limit is defined as usual,

$$Q^2 = -(P_b - P_a)^2 \rightarrow \infty \quad \text{with} \quad x_{Bj} = \frac{Q^2}{2P_a \cdot q} \quad \text{fixed}. \quad (\text{A.16})$$

The mass  $M_a$  of the target  $A$  is kept fixed, while the mass of the produced bound state grows with  $Q$ ,

$$M_b^2 = (P_a + q)^2 \simeq Q^2 \left( \frac{1}{x_{Bj}} - 1 \right). \quad (\text{A.17})$$

In the Breit frame,

$$\begin{aligned} q &= (0, -Q), & k_1 &= \tfrac{1}{2}Q(1, -1), & k_2 &= \tfrac{1}{2}Q(1, 1), \\ P_a &= \frac{Q}{2x_{Bj}}(1, 1), & P_b &= \frac{Q}{2x_{Bj}}(1, 1 - 2x_{Bj}). \end{aligned} \quad (\text{A.18})$$

Using the expression (5.13) for the bound state vertex we find

$$\mathcal{M}(eA \rightarrow eB) = -e^2 \frac{2m_e}{Q} F_{AB}(Q^2). \quad (\text{A.19})$$

The kinematic factors in the expression (A.3) of the cross section are

$$s = \frac{Q^2}{x_{Bj}} \quad \text{and} \quad \epsilon_{\mu\nu} k_2^\mu P_b^\nu = \tfrac{1}{2}Q^2. \quad (\text{A.20})$$

Hence the DIS cross section is

$$\sigma_{\text{scat}}(eA \rightarrow eB) = e^4 \frac{m_e^2}{Q^6} x_{Bj} |F_{AB}(Q^2)|^2 \equiv 2\pi e^2 \frac{d\sigma_{\text{scat}}}{dM_b^2}, \quad (\text{A.21})$$

where the second equality implies an average over the bound state peaks, whose separation in  $M_b^2$  is  $2\pi e^2$  according to (3.19). Converting  $dM_b^2 = -Q^2 dx_{Bj}/x_{Bj}^2$  we find the parton distribution  $f(x_{Bj})$  of the target as

$$\begin{aligned} \frac{d\sigma_{\text{scat}}}{dx_{Bj}} &= \frac{e^2 m_e^2}{2\pi x_{Bj} Q^4} |F_{AB}(Q^2)|^2 \equiv \hat{\sigma}_{\text{scat}}(e\mu \rightarrow e\mu) f(x_{Bj}), \\ f(x_{Bj}) &= \frac{1}{8\pi e^2 m^2} \frac{1}{x_{Bj}} |Q^2 F_{AB}(Q^2)|^2, \end{aligned} \quad (\text{A.22})$$

where  $m$  is the mass of the struck fermion in the target  $A$ .

## Appendix B: Details on the $Q^2 \rightarrow \infty$ limit

The leading  $\mathcal{O}(Q^{-2})$  result (6.7), (6.12) for the DIS form factor  $F_{AB}(Q^2)$  in the Bj limit was found by calculating the contribution to the integral (5.14) for  $x \sim 1/Q$  explicitly. Leading contributions can, in principle, arise also for larger values of  $x$ , if the oscillations of the wave functions cancel the Fourier phase such that a stationary phase arises. In this Appendix we check that such extra contributions are absent.

The form factor was given in (5.14) and becomes

$$F_{AB}(Q^2) = -4i \frac{1 - \eta_a \eta_b}{Q} \int_0^\infty dx \sin\left(\frac{Qx}{2}\right) \left[ \Phi_{0B}^*(x) \Phi_{0A}(x) + \Phi_{1B}^*(x) \Phi_{1A}(x) \left(1 + \frac{4m^2}{\sigma_a \sigma_b} \tilde{\Pi}_a \cdot \Pi_b\right) \right] \quad (\text{B.1})$$

in the Breit frame. It is necessary to check the behavior of the expression in the square brackets of (B.1) by using the asymptotic expressions for  $\Phi_j$ . The wave functions depend on  $x$  through the variables  $\sigma_{a,b}$ . The variable corresponding to the final state is

$$\sigma_b \simeq Q^2 \left( \frac{1}{x_{Bj}} - 1 \right) - \frac{Q}{2x_{Bj}} |x| + \frac{1}{4} x^2, \quad (\text{B.2})$$

which is large for  $x \geq 0$  except very close to the roots

$$x_\pm = Q \left[ \frac{1}{x_{Bj}} \pm \left( \frac{1}{x_{Bj}} - 2 \right) \right]. \quad (\text{B.3})$$

The asymptotic expansion of  $\Phi_B$  can thus be used unless  $|x - x_\pm| \lesssim 1/Q$ . The target wave function depends on

$$\sigma_a = M^2 - \frac{Q}{2x_{Bj}} |x| + \frac{1}{4} x^2. \quad (\text{B.4})$$

The asymptotic formulas for  $\Phi_A$  can be used unless  $x \lesssim 1/Q$  (which was already discussed in the main text) or  $|x - 2Q/x_{Bj}| \lesssim 1/Q$ .

We shall discuss the asymptotics of the wave functions as  $\sigma \rightarrow -\infty$ . To next-to-leading order we find [compare to (3.16)]

$$\begin{aligned} \Phi_1(\sigma) &= \sqrt{\frac{2}{\pi}} \frac{N}{m} \sqrt{e^{2\pi m^2} - 1} e^{-\pi m^2} \left\{ \sin \left[ \frac{\sigma}{2} - m^2 \log(-\sigma) + \arg \Gamma(1 + im^2) \right] \right. \\ &\quad \left. + \frac{m^4 \cos \left[ \frac{\sigma}{2} - m^2 \log(-\sigma) + \arg \Gamma(1 + im^2) \right] + m^2 \sin \left[ \frac{\sigma}{2} - m^2 \log(-\sigma) + \arg \Gamma(1 + im^2) \right]}{\sigma} + \mathcal{O}(\sigma^{-2}) \right\}, \\ \Phi_0(\sigma) &= -i \sqrt{\frac{2}{\pi}} \frac{N}{m} \sqrt{e^{2\pi m^2} - 1} e^{-\pi m^2} \left\{ \cos \left[ \frac{\sigma}{2} - m^2 \log(-\sigma) + \arg \Gamma(1 + im^2) \right] \right. \\ &\quad \left. - \frac{m^4 \sin \left[ \frac{\sigma}{2} - m^2 \log(-\sigma) + \arg \Gamma(1 + im^2) \right] + m^2 \cos \left[ \frac{\sigma}{2} - m^2 \log(-\sigma) + \arg \Gamma(1 + im^2) \right]}{\sigma} + \mathcal{O}(\sigma^{-2}) \right\}. \end{aligned} \quad (\text{B.5})$$

Also, the expression appearing in the last term in the square brackets of (B.1) can be written as

$$\frac{4m^2}{\sigma_a \sigma_b} \tilde{\Pi}_a \cdot \Pi_b = 2m^2 \frac{\sigma_a + \sigma_b + (1 - x_{Bj})^2 Q^2 / x_{Bj}^2}{\sigma_a \sigma_b}. \quad (\text{B.6})$$

Notice that the apparent singularities of this term as  $\sigma_{a,b} \rightarrow 0$  are regularized by the zeroes of  $\Phi_{1A,B}$  in (B.1).

We start by discussing the contributions from the regions where  $x \gtrsim \mathcal{O}(Q^0)$  and the asymptotic formulas (B.5) work. Then the  $\sigma_{a,b}$  are  $\mathcal{O}(Q)$  or larger. Thus the factor (B.6) is suppressed by  $1/Q$ , at least. Neglecting this factor, let us first consider the leading terms of the asymptotic expansion (B.5). The leading terms in (B.1) then combine, through  $\sin(\alpha_a) \sin(\alpha_b) + \cos(\alpha_a) \cos(\alpha_b) = \cos(\alpha_a - \alpha_b)$ , to give

$$F_{AB}(Q^2) \sim -i \frac{1 - \eta_a \eta_b}{Q} \int dx \sin\left(\frac{Qx}{2}\right) \cos \left[ \frac{\sigma_a(x)}{2} - \frac{\sigma_b(x)}{2} + m^2 \log \frac{\sigma_b(x)}{\sigma_a(x)} \right] \quad (\text{B.7})$$

where, for  $P_a^0 = P_b^0 = E$ ,

$$\sigma_a(x) - \sigma_b(x) = M_a^2 - M_b^2 \quad (\text{B.8})$$

is fixed and  $x$  independent. The remaining  $x$  dependence is logarithmic and cannot cancel the rapidly oscillating Fourier phase<sup>17</sup>. Hence no stationary phase can arise from the leading behavior of the first two terms in the integrand, which suggests that the integral is limited to  $xQ$  of  $\mathcal{O}(1)$ . Notice also that we used the approximation of (6.4) and (6.5) in the main text, which is valid when  $x \sim 1/Q$ . The variation of (B.7) due to this approximation is

$$\sim \frac{1}{Q} \int dx \sin\left(\frac{Qx}{2}\right) \frac{x}{Q} \sim \frac{1}{Q^4} \quad (\text{B.9})$$

and thus indeed subleading.

Let us then discuss the inclusion of the next-to-leading terms in (B.5) or the term proportional to the factor (B.6). These terms are suppressed by at least one power of  $\sigma_a$  or  $\sigma_b$ , *i.e.*, a power of  $1/Q$ . As the integral in (B.1) is only multiplied by  $1/Q$ , they could still contribute at  $\mathcal{O}(Q^{-2})$  to the form factor if a suitable stationary phase arises. A stationary phase at a generic  $x \sim Q$  is not relevant, because then  $\sigma_{a,b} \sim Q^2$ , which leads to a suppression of at least  $Q^{-3}$ . As we shall see below, the stationary phases only occur for a range of  $x$  with length  $\mathcal{O}(Q^0)$ , so additional powers of  $Q$  cannot arise from the integration. A stationary phase at  $x \sim Q^0$  would, however, lead to an extra contribution to the form factor at leading order<sup>18</sup>.

Using the expressions (B.5) and (B.6) to expand the factor in the square brackets of (B.1) to next-to-leading order in  $1/\sigma$ , we observe that all possible combinations of the Fourier phase and the phases of  $\Phi_{A,B}$  appear. Neglecting constant and slowly varying factors, they are proportional to  $Qx \pm \sigma_a \pm \sigma_b$ , where the signs of  $\sigma_a$  and  $\sigma_b$  can be different. If they are, however, the situation is as in the case of the leading order analysis in (B.7), and no stationary phases arise. Therefore, we discuss only the phases  $Qx \pm (\sigma_a + \sigma_b)$ , which were absent at leading order. The locations of the stationary phases are found by solving

$$\frac{d}{dx} [Qx \pm (\sigma_a + \sigma_b)] = 0. \quad (\text{B.10})$$

The solutions are given by

$$x \simeq Q \left( \frac{1}{x_{Bj}} \pm 1 \right) \equiv \hat{x}_{\pm}, \quad (\text{B.11})$$

*i.e.*, they occur at generic  $\mathcal{O}(Q)$  values. Near these points the phases behave as  $\sim \frac{1}{4}(x - \hat{x}_{\pm})^2$ , such that the stationary phases are limited to regions having lengths  $\sim Q^0$ , as expected. We conclude that the next-to-leading terms do not contribute to the form factor at leading order<sup>19</sup>.

Finally, let us discuss the contributions to  $F_{AB}(Q^2)$  from the regions where the asymptotic formulas do not hold for either of the wave functions  $\Phi_{A,B}$  [so that  $\sigma_a = \mathcal{O}(Q^0)$  or  $\sigma_b = \mathcal{O}(Q^0)$ ]. The case  $x \sim 1/Q$  was discussed in the main text. The other regions are the neighborhoods of  $x = x_{\pm}$  in (B.3) and  $x = 2Q/x_{Bj}$ , respectively. Let us discuss, for definiteness,  $x = 2Q/x_{Bj}$ . Near this point the asymptotic formulas for  $\Phi_B$  are valid, and we can write down an expression similar to (6.7) and (6.12),

$$F_{AB}(Q^2) \sim \frac{1}{Q} \int_{|x - 2Q/x_{Bj}| \sim 1/Q} dx \sin\left(\frac{Qx}{2}\right) [\dots], \quad (\text{B.12})$$

where  $\dots$  stands for a function of  $xQ$  which is nontrivial but regular within the region of integration. Shifting the integration variable we obtain

$$F_{AB}(Q^2) \sim \frac{1}{Q} \int_{|x| \sim 1/Q} dx \sin\left(\frac{Qx}{2} + \frac{Q^2}{x_{Bj}}\right) [\dots] \sim \frac{1}{Q^2} \int_{|v| \sim 1} dv \sin\left(v + \frac{Q^2}{x_{Bj}}\right) [\dots]. \quad (\text{B.13})$$

The contribution from this region has thus the leading power behavior  $\sim 1/Q^2$ , but also involves the large phase factor  $Q^2/x_{Bj}$ . We interpret that the basically arbitrary phase averages the result to zero. Analogous results are found in the neighborhoods of  $x = x_{\pm}$ .

<sup>17</sup> Notice that this argument does not work when  $\sigma_a$  or  $\sigma_b$  is close to zero, and the logarithmic term varies rapidly. This happens, however, only in the regions where the asymptotic expansions are not reliable to start with.

<sup>18</sup> Stationary phases near the roots  $x_{\pm}$  of (B.3) or near  $x = 2Q/x_{Bj}$  would also be special.

<sup>19</sup> Notice that if  $1 - x_{Bj} \sim Q^{-2}$  so that the final state has a finite mass, one of the stationary phases moves to  $x \sim Q^0$  signaling the breakdown of the results (6.7) and (6.12).

### Appendix C: Asymptotics of the parton distribution at small $x_{Bj}$

It is possible to calculate the  $x_{Bj} \rightarrow 0$  limit of the parton distribution analytically. We again assume that  $\eta_a = -\eta_b = +1$  and start by separating the two contributions in Eq. (6.7) as

$$Q^2 F_{AB}(\eta_a = +) = J_1 + J_2, \quad (C.1)$$

$$J_1 = -8i\sqrt{2\pi} \int_0^\infty dv \sin v \left[ \cos\left(\frac{v}{2x_{Bj}}\right) i\Phi_{0A}(\tau_a) - \sin\left(\frac{v}{2x_{Bj}}\right) \Phi_{1A}(\tau_a) \right], \quad (C.2)$$

$$J_2 = \frac{16im^2\sqrt{2\pi}}{x_{Bj}} \int_0^\infty dv \sin v \sin\left(\frac{v}{2x_{Bj}}\right) \frac{\Phi_{1A}(\tau_a)}{\tau_a}, \quad (C.3)$$

where  $\tau_a$  in (6.4) is a function of  $v/x_{Bj}$ . It would seem that the expansions can be calculated by substituting  $v \rightarrow x_{Bj}v$  in each integral and then developing the factor  $\sin(x_{Bj}v)$  as a series at  $x_{Bj} = 0$ . This, however, leads to integrals that are divergent for  $v \rightarrow \infty$ . Instead we expand the wave functions up to next-to-leading order for  $\tau_a \rightarrow -\infty$ ,

$$\begin{aligned} \Phi_0(\tau_a) &= -\frac{i\sqrt{2\pi}}{2} e^{-\pi m^2} \cos\left(\mu(v) - \frac{v}{2x_{Bj}}\right) \\ &\quad - \frac{im^2\sqrt{2\pi}}{2} e^{-\pi m^2} \left[ \cos\left(\mu(v) - \frac{v}{2x_{Bj}}\right) + (m^2 - M^2) \sin\left(\mu(v) - \frac{v}{2x_{Bj}}\right) \right] \frac{x_{Bj}}{v} + \mathcal{O}\left(\frac{x_{Bj}^2}{v^2}\right), \end{aligned} \quad (C.4)$$

$$\begin{aligned} \Phi_1(\tau_a) &= \frac{\sqrt{2\pi}}{2} e^{-\pi m^2} \sin\left(\mu(v) - \frac{v}{2x_{Bj}}\right) \\ &\quad - \frac{m^2\sqrt{2\pi}}{2} e^{-\pi m^2} \left[ \sin\left(\mu(v) - \frac{v}{2x_{Bj}}\right) + (m^2 - M^2) \cos\left(\mu(v) - \frac{v}{2x_{Bj}}\right) \right] \frac{x_{Bj}}{v} + \mathcal{O}\left(\frac{x_{Bj}^2}{v^2}\right), \end{aligned} \quad (C.5)$$

where  $\mu(v)$  is given in (6.9), and we used  $N$  from (4.16) for the target wave function.

Let us discuss the integral  $J_1$  first. Using the expansions of the wave functions we find

$$J_1 = -8i\pi e^{-\pi m^2} \int_0^\infty dv \sin v \left\{ \cos \mu(v) + m^2 \left[ \cos\left(\mu(v) - \frac{v}{x_{Bj}}\right) + (m^2 - M^2) \sin \mu(v) \right] \frac{x_{Bj}}{v} + \mathcal{O}\left(\frac{x_{Bj}^2}{v^2}\right) \right\}. \quad (C.6)$$

The integral arising from the first term in the wavy brackets was already evaluated in Eq. (6.16). The second term can also be calculated analytically. The first term in the square brackets contains a rapidly oscillating phase as  $x_{Bj} \rightarrow 0$ . Hence its leading contribution arises from the region  $v \sim x_{Bj}$ , and is of  $\mathcal{O}((x_{Bj})^2)$ . The dominant contributions to the second term have  $v \sim (x_{Bj})^0$ , and therefore the result is of  $\mathcal{O}(x_{Bj})$ . The  $\mathcal{O}(x_{Bj}^2/v^2)$  term converges fast enough both as  $v \rightarrow 0$  and as  $v \rightarrow \infty$  for us to develop the sine factor as a series at  $v = 0$  and see that this contribution is of  $\mathcal{O}((x_{Bj})^2)$ . Altogether we get

$$\begin{aligned} J_1 &= -8i\pi e^{-\pi m^2} |\Gamma(1 + im^2)| \left[ \cosh\left(\frac{\pi m^2}{2}\right) \cos\left(\frac{M^2}{2} + m^2 \log x_{Bj}\right) \right. \\ &\quad \left. + x_{Bj} (m^2 - M^2) \sinh\left(\frac{\pi m^2}{2}\right) \sin\left(\frac{M^2}{2} + m^2 \log x_{Bj}\right) \right] + \mathcal{O}((x_{Bj})^2). \end{aligned} \quad (C.7)$$

The integral  $J_2$  can be analyzed similarly. There are, however, complications due to the explicit factor of  $1/x_{Bj}$  appearing in the coefficient in (C.3). First, we need to study the terms of the asymptotic expansion of the integrand up to  $\mathcal{O}(x_{Bj}^2/v^2)$ . This is necessary in order to determine all  $\mathcal{O}(x_{Bj})$  contributions to the form factor from the region of asymptotically high  $v \sim (x_{Bj})^0$ . Second, also the region with small  $v \sim x_{Bj}$ , where the wave function  $\Phi_{1A}$  cannot be estimated in terms of elementary functions, contributes to the form factor at  $\mathcal{O}(x_{Bj})$ . It is hard to find a closed form expression for this contribution, but we will write it down as an integral below.

Let us start with the contributions arising from the region with  $v \sim (x_{Bj})^0$ . Developing the integrand at  $v \rightarrow \infty$  gives

$$\begin{aligned} J_2 &= 8im^2\pi e^{-\pi m^2} \int_0^\infty dv \sin v \left\{ - \left[ \cos\left(\mu(v) - \frac{v}{x_{Bj}}\right) - \cos \mu(v) \right] \frac{1}{v} \right. \\ &\quad \left. + (m^2 - M^2) \left[ \cos\left(\mu(v) - \frac{v}{x_{Bj}}\right) - \cos \mu(v) - m^2 \sin\left(\mu(v) - \frac{v}{x_{Bj}}\right) + m^2 \sin \mu(v) \right] \frac{x_{Bj}}{v^2} + \mathcal{O}\left(\frac{x_{Bj}^2}{v^3}\right) \right\}. \end{aligned} \quad (C.8)$$

The integral over the  $\mathcal{O}(1/v)$  and  $\mathcal{O}(x_{Bj}/v^2)$  terms can be done analytically<sup>20</sup>. This contribution is given by

$$J_{2A} = 8ie^2\pi e^{-\pi m^2} |\Gamma(1+im^2)| \left\{ \sinh\left(\frac{\pi m^2}{2}\right) \cos\left(\frac{M^2}{2} + m^2 \log x_{Bj}\right) - x_{Bj} e^{-\pi m^2/2} \left[ (2m^2 - M^2) \sin\left(\frac{M^2}{2}\right) + m^2(m^2 - M^2) \cos\left(\frac{M^2}{2}\right) \right] + x_{Bj}(m^2 - M^2) \cosh\left(\frac{\pi m^2}{2}\right) \sin\left(\frac{M^2}{2} + m^2 \log x_{Bj}\right) \right\} + \mathcal{O}((x_{Bj})^2). \quad (\text{C.9})$$

The remaining contribution comes from small  $v \sim x_{Bj}$ <sup>21</sup>. It can be isolated by subtracting from the integrand its leading terms given in (C.8), which leads to

$$J_{2B} = \frac{8i\pi m^2}{x_{Bj}} \int_0^\infty dv \sin v \left\{ \sqrt{\frac{2}{\pi}} \sin\left(\frac{v}{2x_{Bj}}\right) \frac{\Phi_{1A}(\tau_a)}{\tau_a} + e^{-\pi m^2} \left[ \cos\left(\mu(v) - \frac{v}{x_{Bj}}\right) - \cos \mu(v) \right] \frac{x_{Bj}}{v} - (m^2 - M^2) e^{-\pi m^2} \left[ \cos\left(\mu(v) - \frac{v}{x_{Bj}}\right) - \cos \mu(v) - m^2 \sin\left(\mu(v) - \frac{v}{x_{Bj}}\right) + m^2 \sin \mu(v) \right] \frac{x_{Bj}^2}{v^2} \right\}. \quad (\text{C.10})$$

After scaling the integration variable by  $x_{Bj}$  the integral reads

$$J_{2B} = 8i\pi m^2 \int_0^\infty dv \sin(x_{Bj}v) \left\{ 2\sqrt{\frac{2}{\pi}} \sin\left(\frac{v}{2}\right) \frac{\Phi_{1A}(\tau_a = M^2 - v)}{M^2 - v} + e^{-\pi m^2} \left[ \cos(\tilde{\mu}(v) - v) - \cos \tilde{\mu}(v) \right] \frac{1}{v} - (m^2 - M^2) e^{-\pi m^2} \left[ \cos(\tilde{\mu}(v) - v) - \cos \tilde{\mu}(v) - m^2 \sin(\tilde{\mu}(v) - v) + m^2 \sin \tilde{\mu}(v) \right] \frac{1}{v^2} \right\}, \quad (\text{C.11})$$

where  $\tilde{\mu}(v) = \mu(x_{Bj}v) = M^2/2 - m^2 \log v$ , and therefore the whole expression in the curly brackets is independent of  $x_{Bj}$ . The leading term as  $x_{Bj} \rightarrow 0$  is then found by expanding the factor  $\sin(x_{Bj}v)$  at small  $x_{Bj}$  and  $v$ ,

$$J_{2B} = iC(m, M)x_{Bj} + \mathcal{O}((x_{Bj})^2), \quad (\text{C.12})$$

where

$$C(m, M) = 8\pi m^2 \int_0^\infty dv \left\{ 2\sqrt{\frac{2}{\pi}} v \sin\left(\frac{v}{2}\right) \frac{\Phi_{1A}(M^2 - v)}{M^2 - v} + e^{-\pi m^2} \left[ \cos(\tilde{\mu}(v) - v) - \cos \tilde{\mu}(v) \right] - (m^2 - M^2) e^{-\pi m^2} \left[ \cos(\tilde{\mu}(v) - v) - \cos \tilde{\mu}(v) - m^2 \sin(\tilde{\mu}(v) - v) + m^2 \sin \tilde{\mu}(v) \right] \frac{1}{v} \right\}. \quad (\text{C.13})$$

Collecting the results,

$$\begin{aligned} Q^2 F_{AB}(\eta_a = +) &= J_1 + J_{2A} + J_{2B} \\ &= -8i\pi e^{-3\pi m^2/2} |\Gamma(1+im^2)| \left\{ \cos\left(\frac{M^2}{2} + m^2 \log x_{Bj}\right) - x_{Bj}(m^2 - M^2) \sin\left(\frac{M^2}{2} + m^2 \log x_{Bj}\right) + x_{Bj} \left[ (2m^2 - M^2) \sin\left(\frac{M^2}{2}\right) + m^2(m^2 - M^2) \cos\left(\frac{M^2}{2}\right) \right] \right\} + iC(m, M)x_{Bj} + \mathcal{O}((x_{Bj})^2). \end{aligned} \quad (\text{C.14})$$

The  $x_{Bj} \rightarrow 0$  expansion for  $x_{Bj}f(x_{Bj})$  obtained by using this formula in (6.1) is shown by the dashed red curves in Fig. 8. Notice that the contribution from the asymptotic oscillations is suppressed by the factor  $e^{-3\pi m^2/2}$  in the non-relativistic limit  $m \rightarrow \infty$ . Such a suppression factor is absent in  $C(m, M)$  of (C.13), which mostly arises from small  $v \sim x_{Bj}$ , *i.e.*, the region where the wave function  $\Phi_{1A}$  is non-vanishing in the nonrelativistic limit.

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<sup>20</sup> Notice that the integral over the  $\mathcal{O}(x_{Bj}/v^2)$  term is convergent at  $v \rightarrow 0$  despite the factor of  $1/v^2$  thanks to cancellations in the numerator.

<sup>21</sup> To be precise, Eq. (C.9) already contains some  $\mathcal{O}(x_{Bj})$  contributions from the region  $v \sim x_{Bj}$ . These are the terms without logarithmic phases.



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